VOLUME 4
JANUARY, 1932 REVIEWS OF MODERN PHYSICS

QUANTUM THEORY OF RADIATION*

By Enrico Fermi
University of Rome, Italy

TABLE OF CONTENTS

Introduction
Part I. Dirac's Theory of Radiation
§1. Fundamental concept.....
§2. Analytic representation.....
§3. Electromagnetic energy of radiation
§4. Hamiltonian of the atom and the r
§5. Classical treatment.....
§6. Perturbation theory.....
§7. Quantum mechanical treatment.
§8. Emission from an excited atom..
§9. Propagation of light in vacuum.
§10. Theory of the Lippman fringes.
§11. Theory of the Doppler effect...
§12. Scattering of radiation from fre
Part II. Theory of Radiation and Dirac's '
§13. Dirac's wave function of the e
§14. Radiation theory in nonrelati
§15. Dirac's theory and scattering
§16. Radiative transitions from pc
Part III. Quantum Electrodynamics..
Bibliography.....

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* Lectures delivered at th
of 1930 at the University of

The renormalizatio Kondo problem*†

Kenneth G. Wilson
Laboratory of Nuclear Studies, C

This review covers several
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"Block spin" methods, ap
explained in Sec. VI. The
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others.

CONTENTS

Introduction
I. Definition of a Renormalization Grou
II. Fixed Points in Perturbation Theory
III. Linearized Behavior Near Fixed Poin
IV. Divergences in Field Theory and S
the Triangle of Renormalization
V. Marginal Operators and Second Order
a Fixed Point
VI. Block Spin Methods: The Two Dime
VII. The Kondo Problem: Introduction an
VIII. Renormalization Group Transforma
Hamiltonian
IX. Impurity Susceptibility and Specific
Hamiltonian
Appendix: Perturbation Expansions for the

INTRODUCTION

One of the most basic themes in th
idea that nature is described locally.
of all physics are local. For example,
specify the behavior of electric and
infinitesimal neighborhood of a point x. It
to specify local equations it is necessary to define
limits, namely the limits which define derivatives. The
of the derivative and the idea of a continuum limit that
underlies the derivative is therefore of great importance in
all of physics.

It is now becoming clear that there is a second form of
continuum limit, called the statistical continuum limit,
which also has a very broad range of applicability through-
out physics. In the statistical continuum limit functions of a
continuous variable are themselves *independent* variables.
For example, the electric and magnetic fields throughout
space can be the independent variables in a statistical
continuum limit. This happens in statistical or quantum
mechanical problems where there are field fluctuations, so
that one has to compute averages over an ensemble of fields.
In statistical calculations one does not compute the field at a

situation where
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* Supported in part by the National Science Foundation.
† This paper is a compilation of material presented as a series of nine lectures at Cargese in Summer 1973.

REVIEWS OF
MODERN PHYSICS
VOLUME 15, NUMBER 1 JANUARY, 19

Stochastic Problems in Physics and Astronomy

S. CHANDRASEKHAR
Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

CONTENTS

INTRODUCTION.....
CHAPTER I. THE PROBLEM OF RANDOM FLIGHTS.....
1. The Simplest One-Dimensional Problem: The Problem of Random Walk.....
2. Random Walk With Reflecting and Absorbing Barriers.....
3. The General Problem of Random Flights: Markoff's Method.....
4. The Solution to the General Problem of Random Flights.....
5. The Passage to a Differential Equation: The Reduction of the Problem of Rand
Flights for Large N to a Boundary Value Problem.....
CHAPTER II. THE THEORY OF THE BROWNIAN MOTION.....
1. Introductory Remarks: Langevin's Equation.....
2. The Theory of the Brownian Motion of a Free Particle.....
3. The Theory of the Brownian Motion of a Particle in a Field of Force. The Harmo
Bound Particle.....
4. The Fokker-Planck Equation. The Generalization of Liouville's Theorem.....
5. General Remarks.....
CHAPTER III. PROBABILITY AFTER-EFFECTS: COLLOID STATISTICS; THE SECON
THERMODYNAMICS. THE THEORY OF COAGULATION, SEDIMENTATION, AND T
OVER POTENTIAL BARRIERS.....
1. The General Theory of Density Fluctuations for Intermittent Observations
Life and the Average Time of Recurrence of a State of Fluctuation.....
2. Experimental Verification of Smoluchowski's Theory: Colloid Statistics.....
3. Probability After-Effects for Continuous Observation.....52
4. On the Reversibility of Thermodynamically Irreversible Processes, the Recurrence of
Improbable States and the Limits of Validity of the Second Law of Thermodynamics.54
5. The Effect of Gravity on Brownian Motion: the Phenomenon of Sedimentation.....57
6. The Theory of Coagulation in Colloids.....59
7. The Escape of Particles over Potential Barriers.....63
CHAPTER IV. PROBABILITY METHODS IN STELLAR DYNAMICS: THE STATISTICS OF THE
GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS.....68
1. Fluctuations in the Force Acting on a Star; the Outline of the Statistical Method.....68
2. The Holtsmark Distribution W(F).....70
3. The Speed of Fluctuations in F.....74

ple examples of self-organization; but we shall see that, even here, the underlying mechanisms are not well understood.

Traditional studies of crystal growth, especially among physicists, have focused primarily on symme-

essentially, but not precisely, identic

[1] I shall make no attempt here to provide of the metallurgical literature. A good st... point for such a survey is the book *Crystal Growth*, edited by B. R. Pamplin. In particular, see the articles on interfacial stability by Delves (1975) and dendrites by Doherty (1975). A more basic reference is Chalmers (1964).

*Research supported in part by AFOSR Grant F44620-76-C-0103.

REVIEWS OF
MODERN PHYSICS
VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN
Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

I. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a third formulation of non-relativistic quantum theory. This formulation was suggested by some of Dirac's[1] remarks concerning the relation of classical action[2] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems A and B interact, the coordinates of one of the systems, say B, may be eliminated from the equations describing the motion of A. The inter-

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (The Clarendon Press, Oxford, 1935), second edition, Section 33; also, Physik. Zeits. Sowjetunion 3, 64 (1933).
[2] P. A. M. Dirac, Rev. Mod. Phys. 17, 195 (1945).

[2] Throughout this paper the term "action" will be used for the time integral of the Lagrangian along a path. When this path is the one actually taken by a particle, moving classically, the integral should more properly be called Hamilton's first principle function.

...sion on a Cay-...
...ension 62
...ercolation
...tion-graph theory and 63
...graphs 63
...-free random graph 64
...formalism 64
...ns and phase transitions 65
...ength 65

...of Mathematics, University of Min-
...esota 55455.

2. ...
F. Alternative a...
...attachment 82
1. Copying mechanism 82
2. Edge redirection 82
3. Walking on a network 83
4. Attaching to edges 83
G. Connection to other problems in statistical mechanics 83
1. The Simon model 83
2. Bose-Einstein condensation 85

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...ss the formation of genuinely com-

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...systems have typically used simple
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...ribed by the Boltzmann transport
...ctories tend at large times to a small
...d limit points, or approach simple
... orbits. In other cases, the trajectories

PREVIEWS OF
MODERN PHYSICS

VOLUME 10 NUMBER 3

MARCH 5, 2019

NIGEL GOLDENFELD

DEPARTMENT OF PHYSICS,
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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Dedication

Pierre Hohenberg

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Leo Kadanoff

http://archives.news.yale.edu/v27.n19/story1.html UIUC Physics https://billdembski.com/friends/remembering-leo-kadanoff/

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REVIEWS OF MODERN PHYSICS, VOLUME 79, JULY–SEPTEMBER 2007

Simple viscous flows: From boundary layers to the renormalization group

John Veysey II and Nigel Goldenfeld

Department of Physics, University of Illinois at Urbana-Champaign,
1110 W. Green Street, Urbana, Illinios 61801, USA

(Published 13 July 2007)

The seemingly simple problem of determining the drag on a body moving through a very viscous fluid has, for over 150 years, been a source of theoretical confusion, mathematical paradoxes, and experimental artifacts, primarily arising from the complex boundary layer structure of the flow near the body and at infinity. The extensive experimental and theoretical literature on this problem is reviewed, with special emphasis on the logical relationship between different approaches. The survey begins with the development of matched asymptotic expansions, and concludes with a discussion of perturbative renormalization-group techniques, adapted from quantum field theory to differential equations. The renormalization-group calculations lead to a new prediction for the drag coefficient, one which can both reproduce and surpass the results of matched asymptotics.

DOI: 10.1103/RevModPhys.79.883 PACS number(s): 47.15.G−, 05.10.Cc, 02.60.Lj, 02.30.Mv

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Citations: 68 Google Scholar; 36 Web of Science

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Womack, PLoS One (2015)

Table 4. Physics top 10 journals by Impact Factor, 2013.

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2	Nature Photonics	18,623	29.958	32.342
3	Advances in Physics	5,026	18.062	27.921
4	Surface Science Reports	4,410	24.562	25.642
5	Physics Reports	21,386	22.910	25.010
6	Nature Physics	20,321	20.603	20.059
7	Nano Today	3,855	18.432	19.202
8	Living Reviews in Relativity	1,600	16.526	18.310
9	Advances in Optics and Photonics	660	9.688	18.194
10	Reports on Progress in Physics	11,421	15.633	16.627

doi:10.1371/journal.pone.0143460.t004

Womack RP (2015) Research Data in Core Journals in Biology, Chemistry, Mathematics, and Physics. PLOS ONE 10(12): e0143460. https://doi.org/10.1371/journal.pone.0143460
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PRL: 2013 IF is 7.728 with **378,568** total citations from **3555** articles published

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When you write a PRL …

- Introductory paragraphs:
  1. General background of the field and state of the art
  2. What has been the specific recent progress, and what is the major question that remains?
  3. “The purpose of this Letter is …”

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  - Journal impact factors
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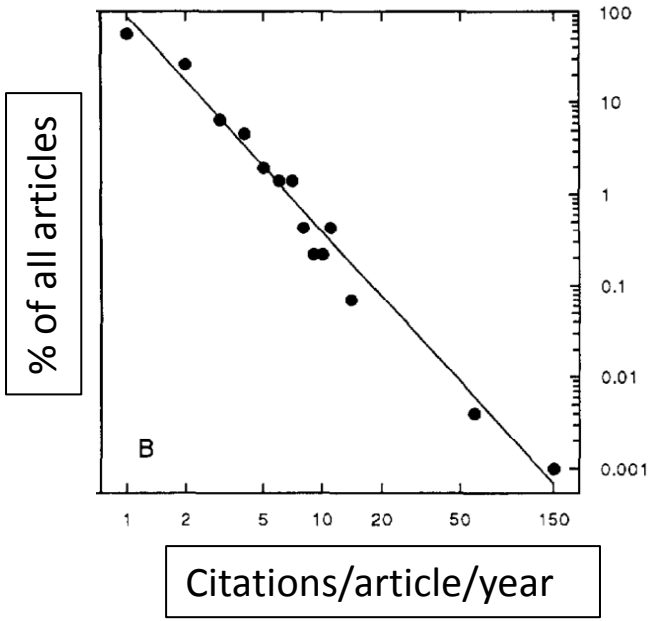

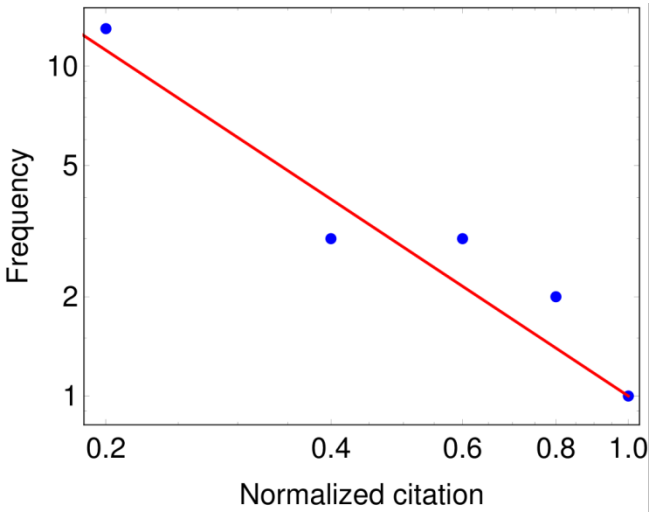

P. Seglen (1992) S. Redner (1998, 2004) H.-Y. Shih

Impact Factor is misleading

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  - Citations within a given journal
  - Citations of a given author
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  - Average citation of articles is 422
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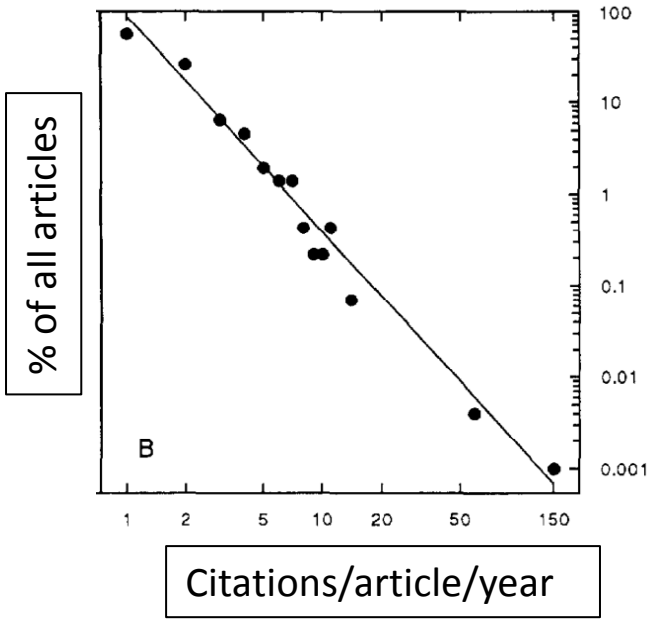

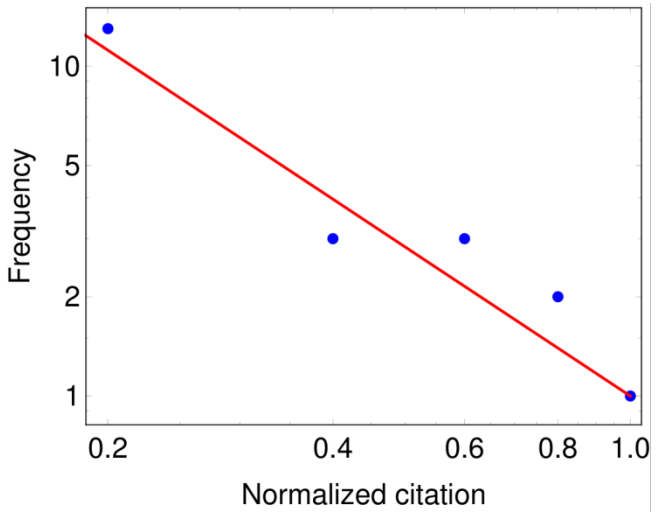

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While impressive, RMP Impact Factor is not at all the whole story

What makes RMP great?

It's not just the sustained high level of erudition, comprehensive scope, clarity and breadth of topics …

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REVIEWS OF
MODERN PHYSICS

VOLUME 15, NUMBER 1 JANUARY, 1943

Stochastic Problems in Physics and Astronomy

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Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

CONTENTS

PAGE

INTRODUCTION ... 2

CHAPTER I. THE PROBLEM OF RANDOM FLIGHTS ... 3

1. The Simplest One-Dimensional Problem: The Problem of Random Walk ... 3
2. Random Walk With Reflecting and Absorbing Barriers ... 5
3. The General Problem of Random Flights: Markoff's Method ... 8
4. The Solution to the General Problem of Random Flights ... 10
5. The Passage to a Differential Equation: The Reduction of the Problem of Random Flights for Large N to a Boundary Value Problem ... 16

CHAPTER II. THE THEORY OF THE BROWNIAN MOTION ... 20

1. Introductory Remarks: Langevin's Equation ... 20
2. The Theory of the Brownian Motion of a Free Particle ... 21
3. The Theory of the Brownian Motion of a Particle in a Field of Force. The Harmonically Bound Particle ... 27
4. The Fokker-Planck Equation. The Generalization of Liouville's Theorem ... 31
5. General Remarks ... 42

CHAPTER III. PROBABILITY AFTER-EFFECTS: COLLOID STATISTICS; THE SECOND LAW OF THERMODYNAMICS. THE THEORY OF COAGULATION, SEDIMENTATION, AND THE ESCAPE OVER POTENTIAL BARRIERS ... 44

1. The General Theory of Density Fluctuations for Intermittent Observations. The Mean Life and the Average Time of Recurrence of a State of Fluctuation ... 44
2. Experimental Verification of Smoluchowski's Theory: Colloid Statistics ... 48
3. Probability After-Effects for Continuous Observation ... 52
4. On the Reversibility of Thermodynamically Irreversible Processes, the Recurrence of Improbable States and the Limits of Validity of the Second Law of Thermodynamics ... 54
5. The Effect of Gravity on Brownian Motion: the Phenomenon of Sedimentation ... 57
6. The Theory of Coagulation in Colloids ... 59
7. The Escape of Particles over Potential Barriers ... 63

CHAPTER IV. PROBABILITY METHODS IN STELLAR DYNAMICS: THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS ... 68

1. Fluctuations in the Force Acting on a Star; the Outline of the Statistical Method ... 68
2. The Holtsmark Distribution $W(F)$ ... 70
3. The Speed of Fluctuations in F ... 74

Theory of dynamic critical phenomena

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and Physik Department, Technische Universität München, 8046, Garching, W. Germany

B. I. Halperin*

Department of Physics, Harvard University, Cambridge, Mass. 02138

An introductory review of the central ideas in the modern theory of dynamic critical phenomena is followed by a more detailed account of recent developments in the field. The concepts of the conventional theory, mode-coupling, scaling, universality, and the renormalization group are introduced and are illustrated in the context of a simple example—the phase separation of a symmetric binary fluid. The renormalization group is then developed in some detail, and applied to a variety of systems. The main dynamic universality classes are identified and characterized. It is found that the mode-coupling and renormalization group theories successfully explain available experimental data at the critical point of pure fluids, and binary mixtures, and at many magnetic phase transitions, but that a number of discrepancies exist with data at the superfluid transition of ^{4}He.

CONTENTS

I. Introduction 436
II. The Symmetric Binary Fluid—A Simple Example 438
A. Hydrodynamics 438
B. Static critical behavior 440
C. Critical dynamics 440
D. The coupled-mode theory 441
III. Basic Definitions and Formalism 442
A. Stochastic models 442
B. Linear and nonlinear hydrodynamics 443
C. Critical behavior 444
1. Static properties 444
2. Dynamic properties 444
D. The dynamic universality classes 445
IV. Renormalization Group for Relaxational Models 445
A. System with no conservation laws: Model A 445
1. The model 445
2. Perturbation theory 446
3. Recursion relations near $d=4$ 446
4. Scaling and universality: stability of the fixed point to all orders in ϵ 448
5. Epsilon and $1/n$ expansions in higher orders 449
6. Related models 450
B. Conserved order parameter: Model B 450
C. Coupling to an auxiliary conserved density: Model C 450
1. The model 450
2. Static properties 451
3. Dynamic properties 451
a. The case $n=1$ 451
b. The case $n>1$ 452
D. Applications to physical systems 452
1. Structural phase transitions 452
a. Simple Hamiltonian model 452
b. Experiments on structural phase transitions 454
2. Magnetic phase transitions 454
a. Hamiltonian examples 454
b. Experimental consequences in magnetic systems 455
3. Order–disorder transition in alloys 455
V. Gas–Liquid and Binary-Fluid Critical Points 455
A. Model H 455
B. Mode-coupling treatment 456
C. Comparison with experiment 457
D. Renormalization group 458
E. Real binary fluids 459
VI. Planar Magnet and Superfluid Helium 459
A. Models E and F 459
B. Dynamic scaling 461
C. Renormalization group 461
D. Comparison with experiment 463
E. Microscopic models 463
F. ^{3}He–^{4}He mixtures and tricritical dynamics 464
G. Two-dimensional superfluid films 465
VII. Heisenberg Magnets 465
A. Antiferromagnet 465
1. Model G 465
2. Critical behavior 466
3. Couplings to other fields and effects of anisotropy 466
4. Experimental studies 466
B. Isotropic ferromagnet 467
1. Model J 467
2. Dynamic scaling and mode coupling 467
3. Renormalization group 468
4. Comparison with experiment 468
VIII. Miscellaneous Topics 468
A. Other dynamic properties 468
1. Sound propagation, EPR, and NMR 468
2. Electrical conductivity 470
B. Nonlinear relaxation 471
C. Other methods of calculation 471
1. Series expansions and computer modeling 471
2. Renormalization group on a lattice 471
3. Cluster dynamics and nucleation theory 472
D. Other systems 472
1. Forces of long but finite range 472
2. Quantum effects and the case $T_c=0$ 472
3. Multicritical points in magnetic systems 473
4. Multicomponent Bose fluid 473
5. Superconductors 473
6. Liquid crystals 474
7. Polymer solutions 474
8. "Phase transitions" far from equilibrium 474
IX. Conclusion 474
A. Summary of results 474
B. Problems and prospects 475
Acknowledgments 475
References 475

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PAGE

INTRODUCTION 2

CHAPTER I. THE PROBLEM OF RANDOM FLIGHTS 3

1. The Simplest One-Dimensional Problem: The Problem of Random Walk 3
2. Random Walk With Reflecting and Absorbing Barriers 5
3. The General Problem of Random Flights: Markoff's Method 8
4. The Solution to the General Problem of Random Flights 10
5. The Passage to a Differential Equation: The Reduction of the Problem of Random Flights for Large N to a Boundary Value Problem 16

CHAPTER II. THE THEORY OF THE BROWNIAN MOTION 20

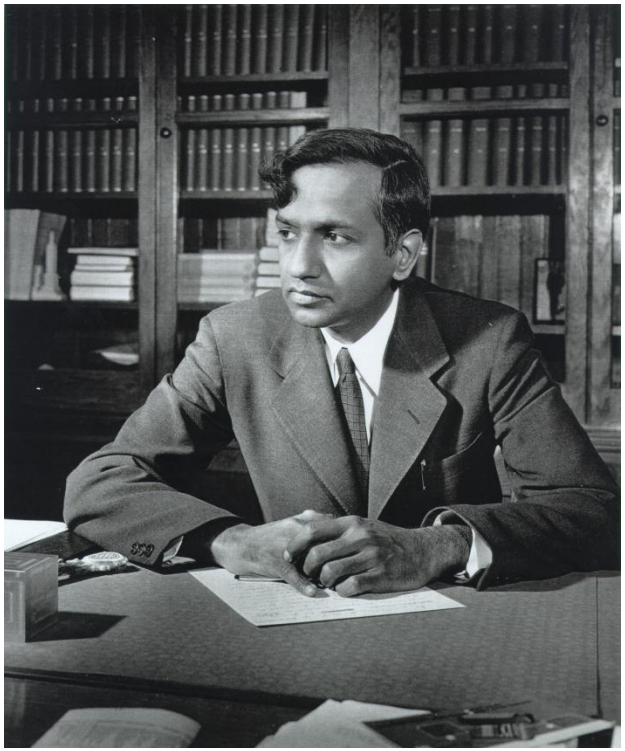
1. Introductory Remarks: Langevin's Equation 20
2. The Theory of the Brownian Motion of a Free Particle 21
3. The Theory of the Brownian Motion of a Particle in a Field of Force. The Harmonically Bound Particle 27
4. The Fokker-Planck Equation. The Generalization of Liouville's Theorem 31
5. General Remarks 42

CHAPTER III. PROBABILITY AFTER-EFFECTS: COLLOID STATISTICS; THE SECOND LAW OF THERMODYNAMICS. THE THEORY OF COAGULATION, SEDIMENTATION, AND THE ESCAPE OVER POTENTIAL BARRIERS 44

1. The General Theory of Density Fluctuations for Intermittent Observations. The Mean Life and the Average Time of Recurrence of a State of Fluctuation 44
2. Experimental Verification of Smoluchowski's Theory: Colloid Statistics 48
3. Probability After-Effects for Continuous Observation 52
4. On the Reversibility of Thermodynamically Irreversible Processes, the Recurrence of Improbable States and the Limits of Validity of the Second Law of Thermodynamics. 54
5. The Effect of Gravity on Brownian Motion: the Phenomenon of Sedimentation 57
6. The Theory of Coagulation in Colloids 59
7. The Escape of Particles over Potential Barriers 63

CHAPTER IV. PROBABILITY METHODS IN STELLAR DYNAMICS: THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS 68

1. Fluctuations in the Force Acting on a Star; the Outline of the Statistical Method 68
2. The Holtsmark Distribution $W(F)$ 70
3. The Speed of Fluctuations in F 74



http://www-news.uchicago.edu/releases/photos/chandra/default.html

CONTENTS

I. Introduction 436
II. The Symmetric Binary Fluid—A Simple Example 438
A. Hydrodynamics 438
B. Static critical behavior 440
C. Critical dynamics 440
441
III. Basic Definitions and Formalism 442
442
B. Linear and nonlinear hydrodynamics 443
C. Critical behavior 444
1. Static properties 444
2. Dynamic properties 444
D. The dynamic universality classes 445
IV. Renormalization Group for Relaxational Models 445
A. System with no conservation laws: Model A 445
1. The model 445
2. Perturbation theory 446
3. Recursion relations near $d=4$ 446
4. Scaling and universality: stability of the fixed point to all orders in ϵ 448
5. Epsilon and $1/n$ expansions in higher orders 449
6. Related models 450
B. Conserved order parameter: Model B 450
C. Coupling to an auxiliary conserved density: Model C 450
1. The model 450
2. Static properties 451
3. Dynamic properties 451
a. The case $n=1$ 451
b. The case $n>1$ 452
D. Applications to physical systems 452
1. Structural phase transitions 452
a. Simple Hamiltonian model 452
b. Experiments on structural phase transitions 454
2. Magnetic phase transitions 454
a. Hamiltonian examples 454
b. Experimental consequences in magnetic systems 455
3. Order–disorder transition in alloys 455
V. Gas–Liquid and Binary-Fluid Critical Points 455
A. Model H 455
B. Mode-coupling treatment 456
C. Comparison with experiment 457
D. Renormalization group 458
E. Real binary fluids 459
VI. Planar Magnet and Superfluid Helium 459
A. Models E and F 459
B. Dynamic scaling 461
C. Renormalization group 461
D. Comparison with experiment 463
E. Microscopic models 463
F. ^{3}He–^{4}He mixtures and tricritical dynamics 464
G. Two-dimensional superfluid films 465
VII. Heisenberg Magnets 465
A. Antiferromagnet 465
1. Model G 465
2. Critical behavior 466
3. Couplings to other fields and effects of anisotropy 466
4. Experimental studies 466
B. Isotropic ferromagnet 467
1. Model J 467
2. Dynamic scaling and mode coupling 467
3. Renormalization group 468
4. Comparison with experiment 468
VIII. Miscellaneous Topics 468
A. Other dynamic properties 468
1. Sound propagation, EPR, and NMR 468
2. Electrical conductivity 470
B. Nonlinear relaxation 471
C. Other methods of calculation 471
1. Series expansions and computer modeling 471
2. Renormalization group on a lattice 471
3. Cluster dynamics and nucleation theory 472
D. Other systems 472
1. Forces of long but finite range 472
2. Quantum effects and the case $T_c=0$ 472
3. Multicritical points in magnetic systems 473
4. Multicomponent Bose fluid 473
5. Superconductors 473
6. Liquid crystals 474
7. Polymer solutions 474
8. "Phase transitions" far from equilibrium 474
IX. Conclusion 474
A. Summary of results 474
B. Problems and prospects 475
Acknowledgments 475
References 475

*Supported in part by the National Science Foundation, under Grant DMR No. 72–02977–AO3.

https://www.physics.harvard.edu/node/728

http://archives.news.yale.edu/v27.n19/story1.html

CONTENTS

I. Introduction 436
II. The Symmetric Binary Fluid—A Simple Example 438
A. Hydrodynamics 438
B. Static critical behavior 440
C. Critical dynamics 440
441
III. Basic Definitions and Formalism 442
442
B. Linear and nonlinear hydrodynamics 443
C. Critical behavior 444
1. Static properties 444
2. Dynamic properties 444
D. The dynamic universality classes 445
445
A. System with no conservation laws: Model A 445
445
2. Perturbation theory 446
3. Recursion relations near $d=4$ 446
4. Scaling and universality: stability of the fixed point to all orders in ϵ 448
5. Epsilon and $1/n$ expansions in higher orders 449
6. Related models 450
B. Conserved order parameter: Model B 450
C. Coupling to an auxiliary conserved density: Model C 450
1. The model 450
2. Static properties 451
3. Dynamic properties 451
a. The case $n=1$ 451
b. The case $n>1$ 452
D. Applications to physical systems 452
1. Structural phase transitions 452
a. Simple Hamiltonian model 452
b. Experiments on structural phase transitions 454
2. Magnetic phase transitions 454
a. Hamiltonian examples 454
b. Experimental consequences in magnetic systems 455
3. Order–disorder transition in alloys 455
V. Gas–Liquid and Binary-Fluid Critical Points 455
A. Model H 455
B. Mode-coupling treatment 456
C. Comparison with experiment 457
D. Renormalization group 458
E. Real binary fluids 459
VI. Planar Magnet and Superfluid Helium 459
A. Models E and F 459
B. Dynamic scaling 461
C. Renormalization group 461
D. Comparison with experiment 463
E. Microscopic models 463
F. ^{3}He–^{4}He mixtures and tricritical dynamics 464
G. Two-dimensional superfluid films 465
VII. Heisenberg Magnets 465
A. Antiferromagnet 465
1. Model G 465
2. Critical behavior 466
3. Couplings to other fields and effects of anisotropy 466
4. Experimental studies 466
B. Isotropic ferromagnet 467
1. Model J 467
2. Dynamic scaling and mode coupling 467
3. Renormalization group 468
4. Comparison with experiment 468
VIII. Miscellaneous Topics 468
A. Other dynamic properties 468
1. Sound propagation, EPR, and NMR 468
2. Electrical conductivity 470
B. Nonlinear relaxation 471
C. Other methods of calculation 471
1. Series expansions and computer modeling 471
2. Renormalization group on a lattice 471
3. Cluster dynamics and nucleation theory 472
D. Other systems 472
1. Forces of long but finite range 472
2. Quantum effects and the case $T_c=0$ 472
3. Multicritical points in magnetic systems 473
4. Multicomponent Bose fluid 473
5. Superconductors 473
6. Liquid crystals 474
7. Polymer solutions 474
8. "Phase transitions" far from equilibrium 474
IX. Conclusion 474
A. Summary of results 474
B. Problems and prospects 475
Acknowledgments 475
References 475

*Supported in part by the National Science Foundation, under Grant DMR No. 72–02977–AO3.

https://www.physics.harvard.edu/node/728

http://archives.news.yale.edu/v27.n19/story1.html

CONTENTS

I. Introduction 436
II. The Symmetric Binary Fluid—A Simple Example 438
A. Hydrodynamics 438
B. Static critical behavior 440
C. Critical dynamics 440
441
III. Basic Definitions and Formalism 442
442
B. Linear and nonlinear hydrodynamics 443
C. Critical behavior 444
1. Static properties 444
2. Dynamic properties 444
D. The dynamic universality classes 445
445
A. System with no conservation laws: Model A 445
445
2. Perturbation theory 446
446
4. Scaling and universality: stability of the fixed point to all orders in ϵ 448
449
6. Related models 450
B. Conserved order parameter: Model B 450
C. Coupling to an auxiliary conserved density: Model C 450
1. The model 450
2. Static properties 451
3. Dynamic properties 451
a. The case $n=1$ 451
b. The case $n>1$ 452
D. Applications to physical systems 452
1. Structural phase transitions 452
a. Simple Hamiltonian model 452
b. Experiments on structural phase transitions 454
2. Magnetic phase transitions 454
a. Hamiltonian examples 454
b. Experimental consequences in magnetic systems 455
3. Order–disorder transition in alloys 455
V. Gas–Liquid and Binary-Fluid Critical Points 455
A. Model H 455
B. Mode-coupling treatment 456
C. Comparison with experiment 457
D. Renormalization group 458
E. Real binary fluids 459
VI. Planar Magnet and Superfluid Helium 459
A. Models E and F 459
B. Dynamic scaling 461
C. Renormalization group 461
D. Comparison with experiment 463
E. Microscopic models 463
F. ^{3}He–^{4}He mixtures and tricritical dynamics 464
G. Two-dimensional superfluid films 465
VII. Heisenberg Magnets 465
A. Antiferromagnet 465
1. Model G 465
2. Critical behavior 466
3. Couplings to other fields and effects of anisotropy 466
4. Experimental studies 466
B. Isotropic ferromagnet 467
1. Model J 467
2. Dynamic scaling and mode coupling 467
3. Renormalization group 468
4. Comparison with experiment 468
VIII. Miscellaneous Topics 468
A. Other dynamic properties 468
1. Sound propagation, EPR, and NMR 468
2. Electrical conductivity 470
B. Nonlinear relaxation 471
C. Other methods of calculation 471
1. Series expansions and computer modeling 471
2. Renormalization group on a lattice 471
3. Cluster dynamics and nucleation theory 472
D. Other systems 472
1. Forces of long but finite range 472
2. Quantum effects and the case $T_c=0$ 472
3. Multicritical points in magnetic systems 473
4. Multicomponent Bose fluid 473
5. Superconductors 473
6. Liquid crystals 474
7. Polymer solutions 474
8. "Phase transitions" far from equilibrium 474
IX. Conclusion 474
A. Summary of results 474
B. Problems and prospects 475
Acknowledgments 475
References 475

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https://www.physics.harvard.edu/node/728

http://archives.news.yale.edu/v27.n19/story1.html

CONTENTS

I. Introduction 436
II. The Symmetric Binary Fluid—A Simple Example 438
A. Hydrodynamics 438
B. Static critical behavior 440
C. Critical dynamics 440
441
III. Basic Definitions and Formalism 442
442
B. Linear and nonlinear hydrodynamics 443
C. Critical behavior 444
1. Static properties 444
2. Dynamic properties 444
D. The dynamic universality classes 445
445
A. System with no conservation laws: Model A 445
445
2. Perturbation theory 446
446
4. Scaling and universality: stability of the fixed point to all orders in ϵ 448
449
6. Related models 450
B. Conserved order parameter: Model B 450
C. Coupling to an auxiliary conserved density: Model C 450
1. The model 450
2. Static properties 451
451
a. The case $n=1$ 451
b. The case $n>1$ 452
452
1. Structural phase transitions 452
a. Simple Hamiltonian model 452
b. Experiments on structural phase transitions 454
2. Magnetic phase transitions 454
a. Hamiltonian examples 454
b. Experimental consequences in magnetic systems 455
3. Order–disorder transition in alloys 455
V. Gas–Liquid and Binary-Fluid Critical Points 455
A. Model H 455
B. Mode-coupling treatment 456
C. Comparison with experiment 457
D. Renormalization group 458
E. Real binary fluids 459
VI. Planar Magnet and Superfluid Helium 459
A. Models E and F 459
B. Dynamic scaling 461
C. Renormalization group 461
D. Comparison with experiment 463
E. Microscopic models 463
F. ^{3}He–^{4}He mixtures and tricritical dynamics 464
G. Two-dimensional superfluid films 465
VII. Heisenberg Magnets 465
A. Antiferromagnet 465
1. Model G 465
2. Critical behavior 466
3. Couplings to other fields and effects of anisotropy 466
4. Experimental studies 466
B. Isotropic ferromagnet 467
1. Model J 467
2. Dynamic scaling and mode coupling 467
3. Renormalization group 468
4. Comparison with experiment 468
VIII. Miscellaneous Topics 468
A. Other dynamic properties 468
1. Sound propagation, EPR, and NMR 468
2. Electrical conductivity 470
B. Nonlinear relaxation 471
C. Other methods of calculation 471
1. Series expansions and computer modeling 471
2. Renormalization group on a lattice 471
3. Cluster dynamics and nucleation theory 472
D. Other systems 472
1. Forces of long but finite range 472
2. Quantum effects and the case $T_c=0$ 472
3. Multicritical points in magnetic systems 473
4. Multicomponent Bose fluid 473
5. Superconductors 473
6. Liquid crystals 474
7. Polymer solutions 474
8. "Phase transitions" far from equilibrium 474
IX. Conclusion 474
A. Summary of results 474
B. Problems and prospects 475
Acknowledgments 475
References 475

*Supported in part by the National Science Foundation, under Grant DMR No. 72–02977–AO3.

https://www.physics.harvard.edu/node/728

http://archives.news.yale.edu/v27.n19/story1.html

Pattern formation outside of equilibrium

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E. Theoretical approaches 953
1. Stochastic models 9[illegible]
a. Stochastic dynamics 953
b. Quenched randomness 954
2. Critical points 954
a. Spatiotemporal intermittency 954
b. Self-organized criticality 955
(i) Definition 955
(ii) Models and mechanisms 955
(iii) Natural phenomena 957
c. Correlation-length inequalities 957
3. Defect-mediated turbulence 958
4. Mean-field dynamics 958
5. Strong turbulence 959
6. Spatial chaos 960
F. Experimental studies 961
VIII. Rayleigh-Bénard Convection 962
A. General features 962
1. Basic equations 962
2. Linear instability and ideal solutions 964
3. Stability balloon 9[illegible]4
4. Amplitude equations 965
5. Phase equations 966
B. Convection models 968
1. Order-parameter models 968
2. [illegible]

IX. Other Fluid Systems 998
A. Convection in fluid mixtures 998
1. Theory 999
a. Basic hydrodynamic equations 999
b. Linear instability 1000
c. Nonlinear states 1000
d. Amplitude equations 1001
e. Other theoretical methods 1001
(i) Lorenz models 1002
(ii) Expansion about the pure fluid 1002
(iii) Expansion about the codimension-two point 1003
(iv) Order-parameter equations 1003
(v) Numerical work 1004
2. Experiment 1005
a. One-dimensional patterns 1005
(i) [illegible] 1005
(ii) Experiments interpretable by the co[illegible]nplex amplitude equation 1005
(α) Rectangular cells 1005
(β) Annular geometry 1006
(iii) Other one-dimensional experiments 1009
b. Two-dimensional patterns 1009
(i) Traveling rolls for $\psi < 0$ 1009
(ii) Squares and rolls for $\psi > 0$ 1010
3. Chaos and noise 1011

What makes RMP great?

My personal view …

Rare, large fluctuations

JANUARY, 1932 REVIEWS OF MODERN PHYSICS VOLUME 4

QUANTUM THEORY OF RADIATION*

By Enrico Fermi

University of Rome, Italy

TABLE OF CONTENTS

Introduction

Part I. Dirac's Theory of Radiation

§1. Fundamental concept
§2. Analytic representation
§3. Electromagnetic energy of radiation
§4. Hamiltonian of the atom and the r
§5. Classical treatment
§6. Perturbation theory
§7. Quantum mechanical treatment
§8. Emission from an excited atom
§9. Propagation of light in vacuum
§10. Theory of the Doppler effect
§11. Theory of the Lippman fringes
§12. Scattering of radiation from fre

Part II. Theory of Radiation and Dirac's

§13. Dirac's wave function of the e
§14. Radiation theory in nonrelati
§15. Dirac's theory and scattering
§16. Radiative transitions from pc

Part III. Quantum Electrodynamics

Bibliography

UNTIL a few years ago radiation which could nomena and the phenomena The first set of phenomena ond set by the theory of li succeeded in constructing a in an unified way both typ the general formulas of Di characteristic examples (P tivistic wave equation of theory of radiation. The general quantum electroc

* Lectures delivered at th of 1930 at the University of

* Supported in part by the National Science Foundation.
† This paper is a compilation of material presented as a series of nine lectures at Cargese in Summer 1973.

Reviews of Modern Physics, Vol. 47, No. 4, October 1975

REVIEWS OF MODERN PHYSICS

VOLUME 15, NUMBER 1 JANUARY, 19

Stochastic Problems in Physics and Astronomy

S. CHANDRASEKHAR

Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

CONTENTS

INTRODUCTION
CHAPTER I. THE PROBLEM OF RANDOM FLIGHTS
1. The Simplest One-Dimensional Problem: The Problem of Random Walk
2. Random Walk With Reflecting and Absorbing Barriers
3. The General Problem of Random Flights: Markoff's Method
4. The Solution to the General Problem of Random Flights
5. The Passage to a Differential Equation: The Reduction of the Problem of Rand Flights for Large N to a Boundary Value Problem

CHAPTER II. THE THEORY OF THE BROWNIAN MOTION
1. Introductory Remarks: Langevin's Equation
2. The Theory of the Brownian Motion of a Free Particle
3. The Theory of the Brownian Motion of a Particle in a Field of Force. The Harmo Bound Particle
4. The Fokker-Planck Equation. The Generalization of Liouville's Theorem
5. General Remarks

CHAPTER III. PROBABILITY AFTER-EFFECTS: COLLOID STATISTICS; THE SECON THERMODYNAMICS. THE THEORY OF COAGULATION, SEDIMENTATION, AND T OVER POTENTIAL BARRIERS
1. The General Theory of Density Fluctuations for Intermittent Observations
Life and the Average Time of Recurrence of a State of Fluctuation
2. Experimental Verification of Smoluchowski's Theory: Colloid Statistics
3. Probability After-Effects for Continuous Observation
4. On the Reversibility of Thermodynamically Irreversible Processes, the Recurrence of Improbable States and the Limits of Validity of the Second Law of Thermodynamics
5. The Effect of Gravity on Brownian Motion: the Phenomenon of Sedimentation
6. The Theory of Coagulation in Colloids
7. The Escape of Particles over Potential Barriers

CHAPTER IV. PROBABILITY METHODS IN STELLAR DYNAMICS: THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS
1. Fluctuations in the Force Acting on a Star; the Outline of the Statistical Method
2. The Holtsmark Distribution $W(F)$
3. The Speed of Fluctuations in F

1

REVIEWS OF MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a third formulation of non-relativistic quantum theory. This formulation was suggested by some of Dirac's[1,2] remarks concerning the relation of classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems A and B interact, the coordinates of one of the systems, say B, may be eliminated from the equations describing the motion of A. The inter-

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (The Clarendon Press, Oxford, 1935), second edition, Section 33; also, Physik. Zeits. Sowjetunion 3, 64 (1933).
[2] P. A. M. Dirac, Rev. Mod. Phys. 17, 195 (1945).
[3] Throughout this paper the term "action" will be used for the time integral of the Lagrangian along a path. When this path is the one actually taken by a particle, moving classically, the integral should more properly be called Hamilton's first principle function.

367

Rare, large fluctuations

The renormalization group: Critical phe
Kondo problem*†
Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 148
This review covers several topics involving renormalizati
solution of the s-wave Kondo Hamiltonian, describing
impurity in a nonmagnetic metal, is explained in detail
"Block spin" methods, applied to the two dimensional
explained in Sec. VI. The first three sections give a re
basic renormalization group ideas, mainly in the con
The relationship of the modern renormalization gro
divergences in statistical mechanics and field theory
renormalization is discussed in Sec. IV. In Sec. V
variables" is discussed in detail, along with the re
renormalization group to its original formulation
others.
CONTENTS
Introduction
I. Definition of a Renormalization Group Transformation
II. Fixed Points in Perturbation Theory
III. Linearized Behavior Near Fixed Points
IV. Divergences in Field Theory and Statistical Mechanic
the Triangle of Renormalization
V. Marginal Operators and Second Order Perturbations Ab
a Fixed Point
VI. Block Spin Methods: The Two Dimensional Ising M
VII. The Kondo Problem: Introduction and Definition of
VIII. Renormalization Group Transformation for the
Hamiltonian
IX. Impurity Susceptibility and Specific Heat for the
Hamiltonian
Appendix: Perturbation Expansions for the Kondo Ha
INTRODUCTION
One of the most basic themes in theoretica
idea that nature is described locally. The
of all physics are local. For example, Max
specify the behavior of electric and magr
infinitesimal neighborhood of a point x. It
to specify local equations it is necessary to define
limits, namely the limits which define derivatives. The
of the derivative and the idea of a continuum limit that
underlies the derivative is therefore of great importance in
all of physics.
It is now becoming clear that there is a second form of
continuum limit, called the statistical continuum limit,
which also has a very broad range of applicability through-
out physics. In the statistical continuum limit functions of a
continuous variable are themselves *independent* variables.
For example, the electric and magnetic fields throughout
space can be the independent variables in a statistical
continuum limit. This happens in statistical or quantum
mechanical problems where there are field fluctuations, so
that one has to compute averages over an ensemble of fields.
In statistical calculations one does not compute the field at a
* Supported in part by the National Science Foundation.
† This paper is a compilation of material presented as a series of
nine lectures at Cargese in Summer 1973.
Reviews of Modern Physics, Vol. 47, No. 4, October 1975
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REVIEWS OF MODERN PHYSICS VOLUME 39, NUMBER 2 APRIL 1967
Static Phenomena Near Critical Points:
Theory and Experiment
LEO P. KADANOFF,* WOLFGANG GÖTZE,† DAVID HAMBLEN, ROBERT HECHT, E. A. S. LEWIS
V. V. PALCIAUSKAS, MARTIN RAYL, J. SWIFT
Department of Physics and Materials Research Laboratory,‡ University of Illinois, Urbana, Illinois
DAVID ASPNES
Department of Physics, Brown University, Providence, Rhode Island
JOSEPH KANE
The Laboratory of Atomic and Solid-State Physics, Cornell University, Ithaca, New York
This paper compares theory and experiment for behavio
are the "critical indices" which describ
These indices are
I. Introduction
II. Molecular Fiel
A. The Order
B. The Landau
C. Landau The
D. Connections
and Thermod
E. The Ising Mo
F. Range of Vali
III. Scaling Laws
A. Formulation
B. Thermodynam
C. Correlation Fu
D. Comparison of
Scaling Law Co
E. Questions About
IV. Magnetic Transition
A. Theoretical Mo
B. Problems in Ferr
C. Experimental Re
V. Classical Liquid-Gas
A. Theory
B. Experiments
C. Use of Inequalities
D. Comparison with
VI. Liquid-Gas-Phase Tra
A. Theory
B. Experiment
C. Summary and Comp
VII. Superfluids
A. Order Parameter for
B. Experimental Result
C. Scaling Law for α_s
D. Perturbation Theory
VIII. Ferroelectric Transitions
References
I. INTRO
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critical points. Several re
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theoretical ideas in this area.
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Instabilities and pattern formation in crystal growth
J. S. Langer
Physics Department and Center for the Joining of Materials, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
Several common modes of crystal growth provide particularly simple and elegant examples
spontaneous pattern formation in nature. Phenomena of interest here are those in which an advan
nonfaceted solidification front suffers an instability and subsequently reorganizes itself into a m
complex mode of behavior. The purpose of this essay is to examine several such situations and, in d
this, to identify a few new theoretical ideas and a larger number of outstanding problems. The syst
studied are those in which solidification is controlled entirely by a single diffusion process, either the
of latent heat away from a moving interface or the analogous redistribution of chemical constituen
Convective effects are ignored, as are most effects of crystalline anisotropy. The linear theory of
Mullins-Sekerka instability is reviewed for simple planar and spherical cases and also for a special mo
of directional solidification. These techniques are then extended to the case of a freely growing dendri
and it is shown how this analysis leads to an understanding of sidebranching and tip-splitting instabiliti
A marginal-stability hypothesis is introduced; and it is argued that this intrinsically nonlinear theory,
valid, permits some to use results of linear-stability analysis to predict dendritic growth rates. The revie
concludes with a discussion of nonlinear effects in directional solidication. The nonplanar, cellular
interfaces which emerge in this situation have much in common with convection patterns in
hydrodynamics. The cellular stability problem is discussed briefly, and some preliminary attempts to de
calculations in the strongly nonlinear regime are summarized.
CONTENTS
I. Introduction 1
II. Phenomena and Models 3
A. Solidification of a pure liquid 3
B. Isothermal solidification of a liquid mixture 5
C. Directional solidification 6
III. The Mullins-Sekerka Instability 8
A. Pure substance: planar interface 8
B. Pure substance: growing sphere 10
C. Directional solidification 11
IV. Dendritic Growth 12
A. The needle crystal 12
B. Dendritic instabilities 15
C. The marginal-stability hypothesis 18
V. Cellular Interfaces 20
A. The one-sided model 20
B. The symmetric model 22
VI. Summary: Outstanding Problems 26
Acknowledgments 27
References 27
I. INTRODUCTION
Outside the realm of biology, some of the most beautiful and familiar examples of spontaneous pattern formation in nature can be found in the growth of crystals. We all have admired snowflakes; and most physicists are aware of the dendritic—that is, tree-like—microstructures which occur during the solidification of alloys. Solidifying systems are extremely interesting for scientific and technological as well as aesthetic reasons. Compared to complex biological processes, these systems represent conceptually simple examples of self-organization; but we shall see that, even here, the underlying mechanisms are not well understood.
Traditional studies of crystal growth, especially among physicists, have focused primarily on symmetries of atomic arrangements, and, more generally, on those properties which are dominated by graphic effects. The formation patterns, however, is an in brium phenomenon which has b sity, mostly by metallurgists w practical way with these phenom materials processes.[1] In this e physicist's interpretation of son on the nonequilibrium problem, this review with a more speculat cent developments. I shall try t in such a way as to emphasize its ber of apparently similar self-or have become fashionable among p biologists, and mathematicians.
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*Research supported in part by AFOSR Grant #44620-76-C-0103.
[1] I shall make no attempt here to provide of the metallurgical literature. A good st ing point for such a survey is the book *Crystal Growth*, edited by B. R. Pamplin. In particular, see the articles on interfacial stability by Delves (1975) and dendrites by Doherty (1975). A more basic reference is Chalmers (1964).
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1

Statistical mechanics of cellular automata
Stephen Wolfram
The Institute for Advanced Study, Princeton, New Jersey 08540
Cellular automata are used as simple mathematical models to investigate self-organization in statistical mechanics. A detailed analysis is given of "elementary" cellular automata consisting of a sequence of sites with values 0 or 1 on a line, with each site evolving deterministically in discrete time steps according to
neighbors. With simple initial configurations, the cellular patterns with fractal dimensions

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002
Statistical mechanics of complex networks
Réka Albert* and Albert-László Barabási
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556
(Published 30 January 2002)
Complex networks describe a wide range of systems in nature and society. Frequently cited examples include the cell, a network of chemicals linked by chemical reactions, and the Internet, a network of routers and computers connected by physical links. While traditionally these systems have been modeled as random graphs, it is increasingly recognized that the topology and evolution of real networks are governed by robust organizing principles. This article reviews the recent advances in the field of complex networks, focusing on the statistical mechanics of network topology and dynamics. After reviewing the empirical data that motivated the recent interest in networks, the authors discuss the main models and analytical tools, covering random graphs, small-world and scale-free networks, the emerging theory of evolving networks, and the interplay between topology and the network's robustness against failures and attacks.
CONTENTS
I. Introduction 48
II. The Topology of Real Networks: Empirical Results 49
A. World Wide Web 49
B. Internet 50
C. Movie actor collaboration network 52
D. Science collaboration graph 52
E. The web of human sexual contacts 52
F. Cellular networks 53
G. Ecological networks 53
H. Phone call network 53
I. Citation networks 53
J. Networks in linguistics 54
K. Power and neural networks 54
L. Protein folding 54
III. Random-Graph Theory 55
A. The Erdős-Rényi model 56
B. Subgraphs 57
C. Graph evolution 58
D. Degree distribution 58
E. Connectedness and diameter 59
F. Clustering coefficient 59
G. Graph spectra 60
IV. Percolation Theory 60
A. Quantities of interest in percolation theory 60
B. General results 61
1. The subcritical phase ($p<p_c$) 61
2. The supercritical phase ($p>p_c$) 62
C. Exact solutions: Percolation on a Cayley tree 62
D. Scaling in the critical region 62
E. Cluster structure 63
F. Infinite-dimensional percolation 63
G. Parallels between random-graph theory and percolation 64
V. Generalized Random Graphs 64
A. Thresholds in a scale-free random graph 65
B. Generating function formalism 65
1. Component sizes and phase transitions 65
2. Average path length 66
C. Random graphs with power-law degree distribution 66
D. Bipartite graphs and the clustering coefficient 67
VI. Small-World Networks 67
A. The Watts-Strogatz model 68
B. Properties of small-world networks 68
1. Average path length 69
2. Clustering coefficient 70
3. Degree distribution 70
4. Spectral properties 71
VII. Scale-Free Networks 71
A. The Barabási-Albert model 71
B. Theoretical approaches 73
C. Limiting cases of the Barabási-Albert model 74
D. Properties of the Barabási-Albert model 74
1. Average path length 75
2. Node degree correlations 75
3. Clustering coefficient 75
4. Spectral properties 76
VIII. The Theory of Evolving Networks 76
A. Preferential attachment $\Pi(k)$ 76
1. Measuring $\Pi(k)$ for real networks 77
2. Nonlinear preferential attachment 77
3. Initial attractiveness 78
B. Growth 78
1. Empirical results 78
2. Analytical results 79
C. Local events 79
1. Internal edges and rewiring 80
2. Internal edges and edge removal 80
D. Growth constraints 81
1. Aging and cost 81
2. Gradual aging 81
E. Competition in evolving networks 82
1. Fitness model 82
2. Edge inheritance 82
F. Alternative mechanisms for preferential attachment 82
1. Copying mechanism 83
2. Edge redirection 83
3. Walking on a network 83
4. Attaching to edges 83
G. Connection to other problems in statistical mechanics 83
1. The Simon model 83
2. Bose-Einstein condensation 85
*Present address: School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.
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47
Reviews of

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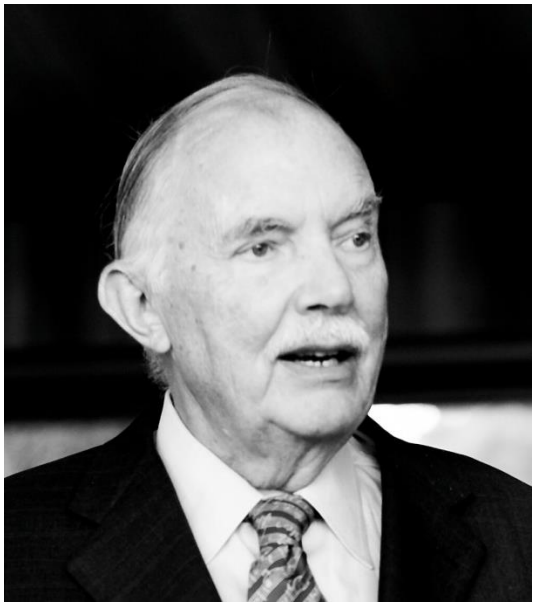
Visionary Editors



John Tate (1929–41, 1947), J. William Buchta (1941–46, 1948–51), Samuel Goudsmit (1951–57), Edward Condon (1957–68), **Lewis Branscomb** (1969–73), **David Pines** (1973–95), George Bertsch (1996–2005), Achim Richter (2006–17), and **Randall Kamien** (2017–)

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- **Lewis Branscomb *et al*. 1969**: “The best papers in the *Reviews of Modern Physics* should be milestones of physics, embodying the intellectual contributions of hundreds of others whose work appears in the original literature” and that *RMP* authors “assume responsibilities: a responsibility to these hundreds of authors whose work may be referenced … and an even greater responsibility to the reader, who is entitled to assume that a paper in *Rev. Mod. Phys*. is as complete, as objective, and as critical as it can reasonably be.”

Editorial Policy

- **Pines *et al*. 1974**: "relax the traditional requirement that a review be *complete*, provided the author has been a major contributor to the field in question".

- **Kamien *et al*. Today**: "Finally, we would like to keep RMP's tradition of publishing articles that provide a *broad **new** framework for understanding the phenomena in a given field*, for example the papers by Feynman, Bethe, and Fermi, but we will be highly selective of such articles"

QUANTUM THEORY OF RADIATION*

By Enrico Fermi
University of Rome, Italy

TABLE OF CONTENTS

Introduction

Part I. Dirac's Theory of Radiation

§1. Fundamental concept 88
§2. Analytic representation 88
§3. Electromagnetic energy of radiation field 90
§4. Hamiltonian of the atom and the radiation field 91
§5. Classical treatment 92
§6. Perturbation theory 93
§7. Quantum mechanical treatment 94
§8. Emission from an excited atom 98
§9. Propagation of light in vacuum 100
§10. Theory of the Lippman fringes 103
§11. Theory of the Doppler effect 105
§12. Scattering of radiation from free electrons 109

Part II. Theory of Radiation and Dirac's Wave Equation

§13. Dirac's wave function of the electron 112
§14. Radiation theory in nonrelativistic approximation 117
§15. Dirac's theory and scattering from free electrons 120
§16. Radiative transitions from positive to negative states 123

Part III. Quantum Electrodynamics 125

Bibliography 132

Introduction

UNTIL a few years ago it had been impossible to construct a theory of radiation which could account satisfactorily both for interference phenomena and the phenomena of emission and absorption of light by matter. The first set of phenomena was interpreted by the wave theory, and the second set by the theory of light quanta. It was not until in 1927 that Dirac succeeded in constructing a quantum theory of radiation which could explain in an unified way both types of phenomena. In this article we shall develop the general formulas of Dirac's theory, and show its applications to several characteristic examples (Part I). In the second part of this work Dirac's relativistic wave equation of the electron will be discussed in relation to the theory of radiation. The third part will be devoted to the problems of the general quantum electrodynamics, and to the difficulties connected with it.

* Lectures delivered at the Symposium for Theoretical Physics during the Summer Session of 1930 at the University of Michigan.

REVIEWS OF
MODERN PHYSICS

Volume 20, Number 2 April, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. Feynman

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a third formulation of non-relativistic quantum theory. This formulation was suggested by some of Dirac's[1,2] remarks concerning the relation of classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems A and B interact, the coordinates of one of the systems, say B, may be eliminated from the equations describing the motion of A. The inter-

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics* (The Clarendon Press, Oxford, 1935), second edition, Section 33; also, Physik. Zeits. Sowjetunion **3**, 64 (1933).
[2] P. A. M. Dirac, Rev. Mod. Phys. **17**, 195 (1945).

[3] Throughout this paper the term "action" will be used for the time integral of the Lagrangian along a path. When this path is the one actually taken by a particle, moving classically, the integral should more properly be called Hamilton's first principle function.

New Perspectives

- RMP provides a summary of a field
- RMP provides a synthesis of a field
- Sometimes RMP presents a **new** perspective, in extreme cases changing the course of the field

REVIEWS OF
MODERN PHYSICS

VOLUME 15, NUMBER 1 JANUARY, 1943

Stochastic Problems in Physics and Astronomy

S. CHANDRASEKHAR
Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

CONTENTS

PAGE

INTRODUCTION 2
CHAPTER I. THE PROBLEM OF RANDOM FLIGHTS 3
1. The Simplest One-Dimensional Problem: The Problem of Random Walk 3
2. Random Walk With Reflecting and Absorbing Barriers 5
3. The General Problem of Random Flights: Markoff's Method 8
4. The Solution to the General Problem of Random Flights 10
5. The Passage to a Differential Equation: The Reduction of the Problem of Random Flights for Large N to a Boundary Value Problem 16

CHAPTER II. THE THEORY OF THE BROWNIAN MOTION 20
1. Introductory Remarks: Langevin's Equation 20
2. The Theory of the Brownian Motion of a Free Particle 21
3. The Theory of the Brownian Motion of a Particle in a Field of Force. The Harmonically Bound Particle 27
4. The Fokker-Planck Equation. The Generalization of Liouville's Theorem 31
5. General Remarks 42

CHAPTER III. PROBABILITY AFTER-EFFECTS: COLLOID STATISTICS; THE SECOND LAW OF THERMODYNAMICS. THE THEORY OF COAGULATION, SEDIMENTATION, AND THE ESCAPE OVER POTENTIAL BARRIERS 44
1. The General Theory of Density Fluctuations for Intermittent Observations. The Mean Life and the Average Time of Recurrence of a State of Fluctuation 44
2. Experimental Verification of Smoluchowski's Theory: Colloid Statistics 48
3. Probability After-Effects for Continuous Observation 52
4. On the Reversibility of Thermodynamically Irreversible Processes, the Recurrence of Improbable States and the Limits of Validity of the Second Law of Thermodynamics 54
5. The Effect of Gravity on Brownian Motion: the Phenomenon of Sedimentation 57
6. The Theory of Coagulation in Colloids 59
7. The Escape of Particles over Potential Barriers 63

CHAPTER IV. PROBABILITY METHODS IN STELLAR DYNAMICS: THE STATISTICS OF THE GRAVITATIONAL FIELD ARISING FROM A RANDOM DISTRIBUTION OF STARS 68
1. Fluctuations in the Force Acting on a Star; the Outline of the Statistical Method 68
2. The Holtsmark Distribution $W(F)$ 70
3. The Speed of Fluctuations in F 74

New Perspectives

- Chandra's famous RMP on Stochastic Processes (9532 GS citations)
- Part of the review concerned the author's ongoing and future work!

Chapter IV

The ideas developed in this chapter are in the main taken from

65. S. Chandrasekhar, Astrophys. J. **94**, 511 (1941).
66. S. Chandrasekhar and J. von Neumann, Astrophys. J. **95**, 489 (1942).
67. S. Chandrasekhar and J. von Neumann, Astrophys. J. **97**, 1, (1943).

§1.—See references 65, 66, and 67; also

68. S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, 1942), Chapters II and V.

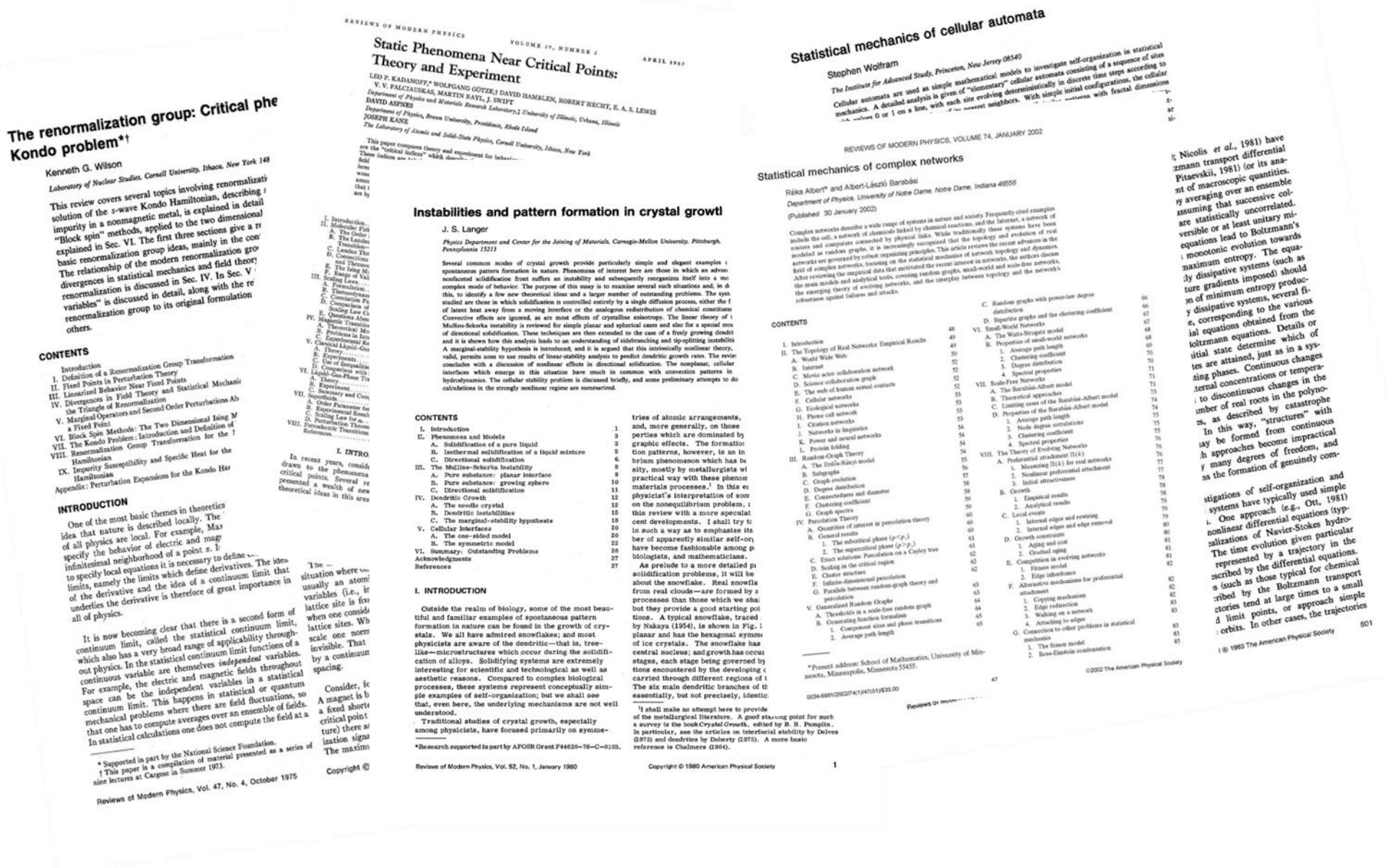
§2.—The problem considered in this section is clearly equivalent to finding the probability of a given *electric* field strength at a point in a gas composed of simple ions. This latter problem was first considered by Holtsmark:

69. J. Holtsmark, Ann. d. Physik **58**, 577 (1919); also Physik. Zeits. **20**, 162 (1919) and **25**, 73 (1924). Among other papers on related subjects we may refer to
70. R. Gans, Ann. d. Physik **66**, 396 (1921).
71. P. Hertz, Math. Ann. **67**, 387 (1909).
72. R. Gans, Physik. Zeits. **23**, 109 (1922).
73. C. V. Raman, Phil. Mag. **47**, 671 (1924).

§3.—See references 66 and 67. See also three further papers on "Dynamical Friction" by Chandrasekhar in forthcoming issues of *The Astrophysical Journal* where further applications of the Fokker-Planck equation will be found.

Previews of Modern Physics

Previews of Modern Physics



The renormalization group: Critical phe
Kondo problem*†

Kenneth G. Wilson

Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 148

This review covers several topics involving renormalizati
solution of the s-wave Kondo Hamiltonian, describing
impurity in a nonmagnetic metal, is explained in detail
"Block spin" methods, applied to the two dimensional
explained in Sec. VI. The first three sections give a re
basic renormalization group ideas, mainly in the con
The relationship of the modern renormalization gro
divergences in statistical mechanics and field theor
renormalization is discussed in Sec. IV. In Sec. V
variables" is discussed in detail, along with the re
renormalization group to its original formulation
others.

CONTENTS

Introduction
I. Definition of a Renormalization Group Transformation
II. Fixed Points in Perturbation Theory
III. Linearized Behavior Near Fixed Points
IV. Divergences in Field Theory and Statistical Mechanic
the Triangle of Renormalization
V. Marginal Operators and Second Order Perturbations Ab
a Fixed Point
VI. Block Spin Methods: The Two Dimensional Ising M
VII. The Kondo Problem: Introduction and Definition of
VIII. Renormalization Group Transformation for the
Hamiltonian
IX. Impurity Susceptibility and Specific Heat for the
Hamiltonian
Appendix: Perturbation Expansions for the Kondo Ha

INTRODUCTION

One of the most basic themes in theoretica
idea that nature is described locally. The
of all physics are local. For example, Max
specify the behavior of electric and magn
infinitesimal neighborhood of a point x. I
to specify local equations it is necessary to define
limits, namely the limits which define derivatives. The idea
of the derivative and the idea of a continuum limit that
underlies the derivative is therefore of great importance in
all of physics.

It is now becoming clear that there is a second form of
continuum limit, called the statistical continuum limit,
which also has a very broad range of applicability through-
out physics. In the statistical continuum limit functions of a
continuous variable are themselves *independent* variables.
For example, the electric and magnetic fields throughout
space can be the independent variables in a statistical
continuum limit. This happens in statistical or quantum
mechanical problems where there are field fluctuations, so
that one has to compute averages over an ensemble of fields.
In statistical calculations one does not compute the field at a

* Supported in part by the National Science Foundation.
† This paper is a compilation of material presented as a series of
nine lectures at Cargese in Summer 1973.

Reviews of Modern Physics, Vol. 47, No. 4, October 1975

REVIEWS OF MODERN PHYSICS VOLUME 39, NUMBER 2 APRIL 1967

Static Phenomena Near Critical Points:
Theory and Experiment

LEO P. KADANOFF,* WOLFGANG GÖTZE,† DAVID HAMBLEN, ROBERT HECHT, E. A. S. LEWIS
V. V. PALCIAUSKAS, MARTIN RAYL, J. SWIFT
DAVID ASPNES
Department of Physics and Materials Research Laboratory,‡ University of Illinois, Urbana, Illinois
JOSEPH KANE
The Laboratory of Atomic and Solid-State Physics, Cornell University, Ithaca, New York

This paper compares theory and experiment for behavio
are the "critical indices" which describ
These indices are

Instabilities and pattern formation in crystal growt

J. S. Langer

Physics Department and Center for the Joining of Materials, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

Several common modes of crystal growth provide particularly simple and elegant examples
spontaneous pattern formation in nature. Phenomena of interest here are those in which an advan
nonfaceted solidification front suffers an instability and subsequently reorganizes itself into a mo
complex mode of behavior. The purpose of this essay is to examine several such situations and, in d
this, to identify a few new theoretical ideas and a larger number of outstanding problems. The syst
studied are those in which solidification is controlled entirely by a single diffusion process, either the f
of latent heat away from a moving interface or the analogous redistribution of chemical constituen
Convective effects are ignored, as are most effects of crystalline anisotropy. The linear theory of t
Mullins-Sekerka instability is reviewed for simple planar and spherical cases and also for a special mo
of directional solidification. These techniques are then extended to the case of a freely growing dendri
and it is shown how this analysis leads to an understanding of sidebranching and tip-splitting instabiliti
A marginal-stability hypothesis is introduced; and it is argued that this intrinsically nonlinear theory,
valid, permits some to use results of linear-stability analysis to predict dendritic growth rates. The revie
concludes with a discussion of nonlinear effects in directional solidification. The nonplanar, cellular
interfaces which emerge in this situation have much in common with convection patterns in
hydrodynamics. The cellular stability problem is discussed briefly, and some preliminary attempts to do
calculations in the strongly nonlinear regime are summarized.

CONTENTS

I. Introduction 1
II. Phenomena and Models 3
A. Solidification of a pure liquid 3
B. Isothermal solidification of a liquid mixture 5
C. Directional solidification 6
III. The Mullins-Sekerka Instability 8
A. Pure substance: planar interface 9
B. Pure substance: growing sphere 10
C. Directional solidification 11
IV. Dendritic Growth 12
A. The needle crystal 12
B. Dendritic instabilities 15
C. The marginal-stability hypothesis 18
V. Cellular Interfaces 20
A. The one-sided model 20
B. The symmetric model 22
VI. Summary: Outstanding Problems 26
Acknowledgments 27
References 27

I. INTRODUCTION

Outside the realm of biology, some of the most beautiful and familiar examples of spontaneous pattern formation in nature can be found in the growth of crystals. We all have admired snowflakes; and most physicists are aware of the dendritic—that is, tree-like—microstructures which occur during the solidification of alloys. Solidifying systems are extremely interesting for scientific and technological as well as aesthetic reasons. Compared to complex biological processes, these systems represent conceptually simple examples of self-organization; but we shall see that, even here, the underlying mechanisms are not well understood.

Traditional studies of crystal growth, especially among physicists, have focused primarily on symmetries of atomic arrangements, and, more generally, on those properties which are dominated by geometric effects. The formation of interesting pattern, however, is an important phenomenon which has been a necessity, mostly by metallurgists who practical way with these phenomena materials processes.[1] In this essay physicist's interpretation of some on the nonequilibrium problem, this review with a more speculative cent developments. I shall try to in such a way as to emphasize its ber of apparently similar self-or have become fashionable among p biologists, and mathematicians.

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*Research supported in part by AFOSR Grant F44620-76-C-0100.

[1] I shall make no attempt here to provide of the metallurgical literature. A good starting point for such a survey is the book *Crystal Growth*, edited by B. R. Pamplin. In particular, see the articles on interfacial stability by Delves (1975) and dendrites by Doherty (1975). A more basic reference is Chalmers (1964).

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1

Statistical mechanics of cellular automata

Stephen Wolfram

The Institute for Advanced Study, Princeton, New Jersey 08540

Cellular automata are used as simple mathematical models to investigate self-organization in statistical mechanics. A detailed analysis is given of "elementary" cellular automata consisting of a sequence of sites with values 0 or 1 on a line, with each site evolving deterministically in discrete time steps according to

Statistical mechanics of complex networks

Réka Albert* and Albert-László Barabási

Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556

(Published 30 January 2002)

Complex networks describe a wide range of systems in nature and society. Frequently cited examples include the cell, a network of chemicals linked by chemical reactions, and the Internet, a network of routers and computers connected by physical links. While traditionally these systems have been modeled as random graphs, it is increasingly recognized that the topology and evolution of real networks are governed by robust organizing principles. This article reviews the recent advances in the field of complex networks, focusing on the statistical mechanics of network topology and dynamics. After reviewing the empirical data that motivated the recent interest in networks, the authors discuss the main models and analytical tools, covering random graphs, small-world and scale-free networks, as the emerging theory of evolving networks, and the interplay between topology and the network's robustness against failures and attacks.

REVIEWS OF MODERN PHYSICS, VOLUME 74, JANUARY 2002

CONTENTS

I. Introduction 48
II. The Topology of Real Networks: Empirical Results 49
A. World Wide Web 49
B. Internet 50
C. Movie actor collaboration network 52
D. Science collaboration graph 52
E. The web of human sexual contacts 52
F. Cellular networks 53
G. Ecological networks 53
H. Phone call network 53
I. Citation networks 54
J. Networks in linguistics 54
K. Power and neural networks 54
L. Protein folding 54
III. Random Graph Theory 55
A. The Erdős-Rényi model 56
B. Subgraphs 57
C. Graph evolution 58
D. Degree distribution 58
E. Connectedness and diameter 59
F. Clustering coefficient 59
G. Graph spectra 60
IV. Percolation Theory 60
A. Quantities of interest in percolation theory 60
B. General results 61
1. The subcritical phase ($p<p_c$) 61
2. The supercritical phase ($p>p_c$) 62
C. Exact solutions: Percolation on a Cayley tree 62
D. Scaling in the critical region 62
E. Cluster structure 63
F. Infinite-dimensional percolation 63
G. Parallels between random-graph theory and percolation 64
V. Generalized Random Graphs 64
A. Thresholds in a scale-free random graph 65
B. Generating function formalism 65
1. Component size and phase transitions 66
2. Average path length 66
C. Random graphs with power-law degree distribution 66
D. Bipartite graphs and the clustering coefficient 67
VI. Small-World Networks 67
A. The Watts-Strogatz model 68
B. Properties of small-world networks 68
1. Average path length 69
2. Clustering coefficient 70
3. Degree distribution 71
4. Spectral properties 71
VII. Scale-Free Networks 71
A. The Barabási-Albert model 73
B. Theoretical approaches 74
C. Limiting cases of the Barabási-Albert model 74
D. Properties of the Barabási-Albert model 75
1. Average path length 75
2. Node degree correlations 75
3. Clustering coefficient 76
4. Spectral properties 76
VIII. The Theory of Evolving Networks 76
A. Preferential attachment [$\Pi(k)$] for real networks 77
1. Measuring $\Pi(k)$ for real networks 77
2. Nonlinear preferential attachment 78
3. Initial attractiveness 78
B. Growth 78
1. Empirical results 79
2. Analytical results 79
C. Local events 79
1. Internal edges and rewiring 80
2. Internal edges and edge removal 80
D. Growth constraints 81
1. Aging and cost 81
2. Gradual aging 81
E. Competition in evolving networks 82
1. Fitness model 82
2. Edge inheritance 82
F. Alternative mechanisms for preferential attachment 82
1. Copying mechanism 83
2. Edge redirection 83
3. Walking on a network 83
4. Attaching to edges 83
G. Connection to other problems in statistical mechanics 83
1. The Simon model 83
2. Bose-Einstein condensation 85

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47

Reviews of Modern Physics

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Nicolis *et al.*, 1981) have
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equations lead to Boltzmann's
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scribed by the differential equations.
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ctories tend at large times to a small
d limit points, or approach simple
orbits. In other cases, the trajectories

601

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Instabilities & Pattern Formation

- Langer's RMP (1980)
  - Citations: 2716 (GS), 1908 (WoS)

Instabilities and pattern formation in crystal growth*

J. S. Langer

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Several common modes of crystal growth provide particularly simple and elegant examples of spontaneous pattern formation in nature. Phenomena of interest here are those in which an advancing nonfaceted solidification front suffers an instability and subsequently reorganizes itself into a more complex mode of behavior. The purpose of this essay is to examine several such situations and, in doing this, to identify a few new theoretical ideas and a larger number of outstanding problems. The systems studied are those in which solidification is controlled entirely by a single diffusion process, either the flow of latent heat away from a moving interface or the analogous redistribution of chemical constituents. Convective effects are ignored, as are most effects of crystalline anisotropy. The linear theory of the Mullins-Sekerka instability is reviewed for simple planar and spherical cases and also for a special model of directional solidification. These techniques are then extended to the case of a freely growing dendrite, and it is shown how this analysis leads to an understanding of sidebranching and tip-splitting instabilities. A marginal-stability hypothesis is introduced; and it is argued that this intrinsically nonlinear theory, if valid, permits aone to use results of linear-stability analysis to predict dendritic growth rates. The review concludes with a discussion of nonlinear effects in directional solidication. The nonplanar, cellular interfaces which emerge in this situation have much in common with convection patterns in hydrodynamics. The cellular stability problem is discussed briefly, and some preliminary attempts to do calculations in the strongly nonlinear regime are summarized.

CONTENTS

I.	Introduction	1
II.	Phenomena and Models	3
	A. Solidification of a pure liquid	3
	B. Isothermal solidification of a liquid mixture	5
	C. Directional solidification	6
III.	The Mullins-Sekerka Instability	8
	A. Pure substance: planar interface	8
	B. Pure substance: growing sphere	10
	C. Directional solidification	11
IV.	Dendritic Growth	12
	A. The needle crystal	12
	B. Dendritic instabilities	15
	C. The marginal-stability hypothesis	18
V.	Cellular Interfaces	20
	A. The one-sided model	20
	B. The symmetric model	22
VI.	Summary: Outstanding Problems	26
Acknowledgments		27
References		27

I. INTRODUCTION

Outside the realm of biology, some of the most beautiful and familiar examples of spontaneous pattern formation in nature can be found in the growth of crystals. We all have admired snowflakes; and most physicists are aware of the dendritic—that is, tree-like—microstructures which occur during the solidification of alloys. Solidifying systems are extremely interesting for scientific and technological as well as aesthetic reasons. Compared to complex biological processes, these systems represent conceptually simple examples of self-organization; but we shall see that, even here, the underlying mechanisms are not well understood.

Traditional studies of crystal growth, especially among physicists, have focused primarily on symmetries of atomic arrangements, surface anisotropies, and, more generally, on those near-equilibrium properties which are dominated by atomic and crystallographic effects. The formation of complex solidification patterns, however, is an intrinsically nonequilibrium phenomenon which has been studied, out of necessity, mostly by metallurgists who must deal in a very practical way with these phenomena in the design of materials processes.[1] In this essay, I shall present a physicist's interpretation of some of the previous work on the nonequilibrium problem, and shall supplement this review with a more speculative discussion of recent developments. I shall try to describe the problem in such a way as to emphasize its relationship to a number of apparently similar self-organizing systems that have become fashionable among physicists, chemists, biologists, and mathematicians.

As prelude to a more detailed presentation of special solidification problems, it will be useful to think a bit about the snowflake. Real snowflakes—those that fall from real clouds—are formed by more complicated processes than those which we shall consider here; but they provide a good starting point for posing questions. A typical snowflake, traced from a photograph by Nakaya (1954), is shown in Fig. 1. The pattern is planar and has the hexagonal symmetry characteristic of ice crystals. The snowflake has grown out from a central nucleus; and growth has occurred in a number of stages, each stage being governed by the external conditions encountered by the developing crystal as it is carried through different regions of the atmosphere. The six main dendritic branches of the crystal are essentially, but not precisely, identical to one another.

*Research supported in part by AFOSR Grant F44620-76-C-0103.

[1] I shall make no attempt here to provide a complete review of the metallurgical literature. A good starting point for such a survey is the book *Crystal Growth*, edited by B. R. Pamplin. In particular, see the articles on interfacial stability by Delves (1975) and dendrites by Doherty (1975). A more basic reference is Chalmers (1964).

https://www.aps.org/programs/honors/prizes/prizerecipient.cfm?first_nm=James&last_nm=Langer&year=1997

Instabilities & Pattern Formation

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Instabilities & Pattern Formation

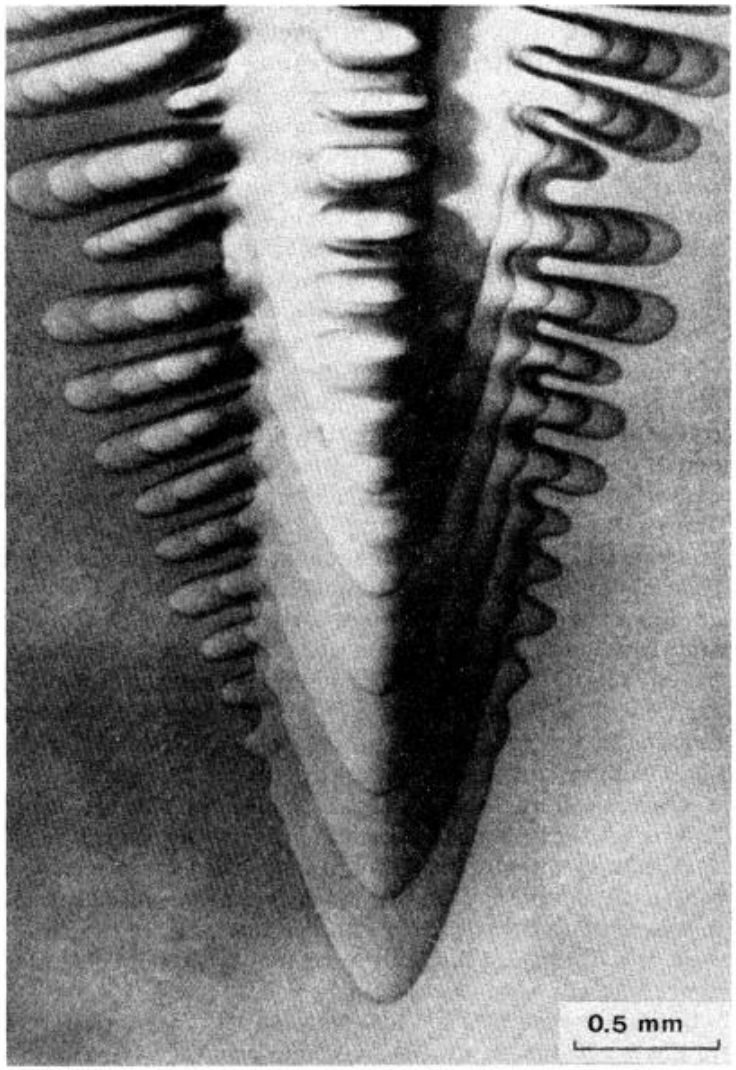
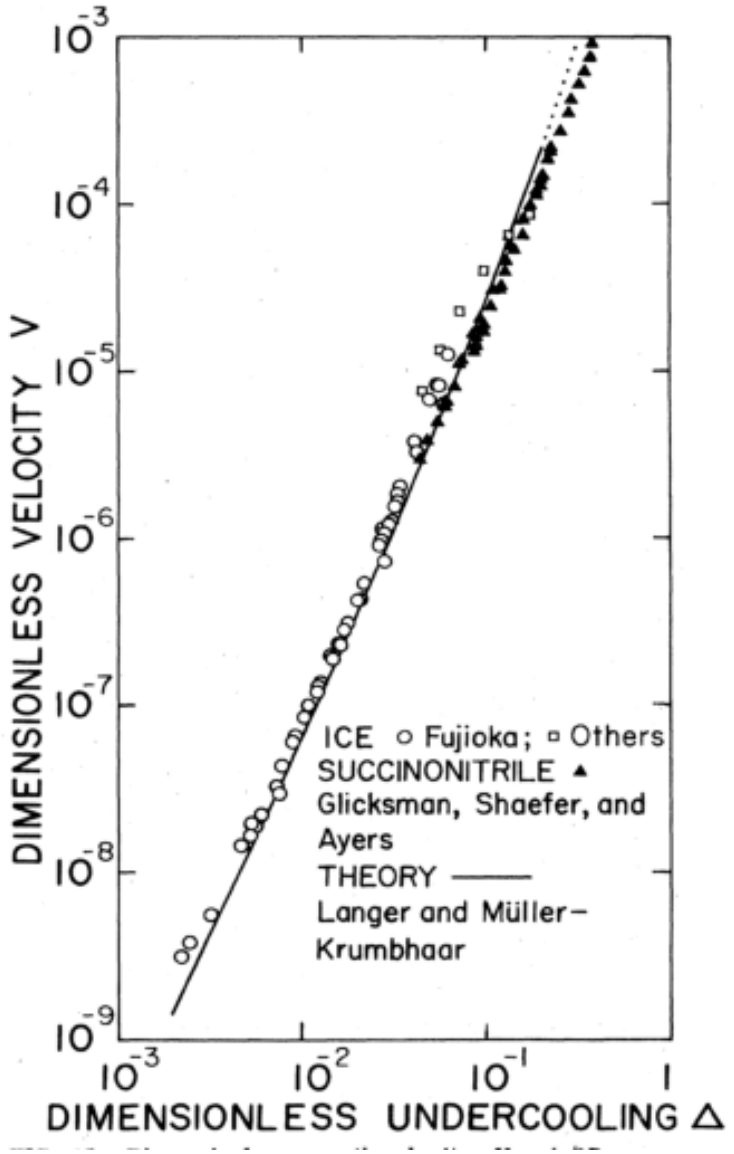


FIG. 12. Multiple-exposure photograph of a downward-growing succinonitrile dendrite (Glicksman, to be published).



Langer (1980)

Surprisingly, Fujioka's data for growth rates of ice dendrites also fit nicely on the $V(\Delta)$ curve in Fig. 15. For ice, β is roughly 4; and the twofold, rather than fourfold, symmetry about the growth axis would seem to require $j = 3$ in (4.35). The resulting value of $\sigma^* \cong 0.025$ seems exactly right; but, in view of the major uncertainties in the theory, I think that this precise agreement must be largely fortuitous. It seems to me that the present form of the theory cannot be adequate to describe so highly anisotropic a system, and that, in particular, anisotropic attachment kinetics must be included in both the steady-state calculation and the stability analysis.

Despite its apparent success in the interpretation of experimental results, the stability theory of dendritic growth is fundamentally incomplete. The stable operating point of a dendrite seems to be a complex, dynamic, essentially nonlinear state of the system; and I must emphasize that we do not understand this state at all yet. We have no satisfactory model of the mechanism by which the tip instability at $\sigma < \sigma^*$ restores σ to the neighborhood of σ^* without destroying the dendritic structure. Presumably, there exists some higher-

Langer (1980)

- These are questions (1)-(3) of a long list of (11) problems.
- In the face of an apparently successful theoretical start, Langer was so dissatisfied with the basis of the work that he outlined a research program that would take another 20 years to begin to come to fruition!

Let us start with questions pertaining to dendrites. The sidebranching mechanism and the marginal-stability hypothesis have provided us with a new starting point for both theoretical and experimental investigation. However:

(1) We have no firm theoretical basis for the marginal-stability hypothesis. Is it correct? If so, what actually happens in the nonlinear operating mode of the dendrite? Does the system undergo a limit cycle? What role is played by thermal fluctuations?

(2) We have no reliable and systematic means for evaluating the stability parameter σ^*. Part of the problem is in the numerical analysis. How does one obtain accurate solutions to a non-Hermitean stability problem, Eq. (4.24), for a semi-infinite system ($0 \leq \xi < \infty$) in which the most interesting deformations—the sidebranches—grow without bound at large ξ? Does the spherical approximation have any validity? How important are capillary corrections to the steady-state solution?

These are the most basic and immediate theoretical problems. Questions of more practical interest include:

(3) What is the effect of crystalline anistropy? How does one include anisotropic attachment kinetics in the steady-state problem or in the stability analysis? Can one construct a quantitative theory for a strongly anisotropic system such as, say, ice?

Instabilities & Pattern Formation

- **Jim Langer**: “Interesting! I had forgotten about that paper. Much of it was wrong, as I guessed in the conclusion. The theory was done correctly with your help in the next five years or so, which must be your main point.”

- “Of course you can quote me. As you know, I’ve always insisted that it’s important to make mistakes.”

APS 2012
In Boston!

Static Critical Phenomena

- Kadanoff's RMP (1967)
  - Citations: 1697 (GS)

REVIEWS OF MODERN PHYSICS VOLUME 39, NUMBER 2 APRIL 1967

Static Phenomena Near Critical Points: Theory and Experiment

LEO P. KADANOFF,* WOLFGANG GÖTZE,† DAVID HAMBLEN, ROBERT HECHT, E. A. S. LEWIS
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This paper compares theory and experiment for behavior very near critical points. The primary experimental results are the "critical indices" which describe singularities in various thermodynamic derivatives and correlation functions. These indices are tabulated and compared with theory. The basic theoretical ideas are introduced via the molecular field approach, which brings in the concept of an order parameter and suggests that there are close relations among different phase transition problems. Although this theory is qualitatively correct it is quantitatively wrong, it predicts the wrong values of the critical indices. Another theoretical approach, the "scaling law" concept, which predicts relations among these indices, is described. The experimental evidence for and against the scaling laws is assessed. It is suggested that the scaling laws provide a promising approach to understanding phenomena near the critical point, but that they are by no means proved or disproved by the existing experimental data.

CONTENTS

I. Introduction 395
II. Molecular Field Theories 396
A. The Order Parameter 396
B. The Landau Theory of the Second-Order Phase Transition—Thermodynamics 396
C. Landau Theory—Fluctuations 398
D. Connections between Correlation Functions and Thermodynamics 400
E. The Ising Model 400
F. Range of Validity of Landau Theory 402
III. Scaling Laws 403
A. Formulation 403
B. Thermodynamic Results 404
C. Correlation Function Results 405
D. Comparison of Ising Model Results with Scaling Law Conclusions 405
E. Questions About Real Phase Transitions 406
IV. Magnetic Transitions 406
A. Theoretical Models 406
B. Problems in Interpreting Experimental Data 407
C. Experimental Results 408
V. Classical Liquid–Gas Transitions 417
A. Theory 417
B. Experiments 418
C. Use of Inequalities 422
D. Comparison with Scaling Laws 423
VI. Liquid–Gas-Phase Transition in Quantum Liquids 423
A. Theory 423
B. Experiment 424
C. Summary and Comparison with Classical Fluids 427
VII. Superfluids 427
A. Order Parameter for this Transition 427
B. Experimental Results 427
C. Scaling Law for ρ_s 427
D. Perturbation Theoretic Approach 428
VIII. Ferroelectric Transitions 428
References 429

I. INTRODUCTION

In recent years, considerable attention has been drawn to the phenomena which occur very near critical points. Several recent conferences[1,2] have presented a wealth of new experimental data and theoretical ideas in this area. These conferences have broadcast the fact that there are quite marked similarities between apparently very different phase transitions. An antiferromagnet near its Néel point behaves quite similarly to a liquid near its critical point. The superconducting transition is not very different from several ferroelectric transitions. In all cases, there is an apparently rather simple behavior in the region right around the critical point.

This simplicity and similarity among phase transitions is not fully elucidated theoretically. Some of the qualitative features of this behavior are reasonably well understood; others remain a complete mystery.

In this paper we review the present status of theory and experiment in this area, concentrating on the time-independent properties of systems near T_c. Thus, we look at thermodynamic derivatives and time-independent correlations but ignore the very interesting work on transport coefficients and time-dependent correlations. The particular subject is what can be learned by comparing different phase transitions with each other and with the existing theories. How are different phase transitions alike? In what ways do they differ? Why should we expect these similarities and differences?

Because we are considering such a broad range of phenomena, we cannot expect our readers to be experts in any particular area we describe. Consequently, we attempt to provide explanations and discussions which will be comprehensible to the nonexpert. We are hopeful that our treatment will provide some picture of the interrelations within this broad field.

In the next section, the important theoretical ideas—the order parameter, the choice between different phases, long-range correlations, and fluctuations—are introduced via the molecular field approximation. These results are tested by comparing them with

Leo’s great idea

Coarse-graining the Hamiltonian

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SCALING LAWS FOR ISING MODELS NEAR T_c*

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Department of Physics, University of Illinois
Urbana, Illinois

(*Received* 3 *February* 1966)

Abstract

A model for describing the behavior of Ising models very near T_c is introduced. The description is based upon dividing the Ising model into cells which are microscopically large but much smaller than the coherence length and then using the total magnetization within each cell as a collective variable. The resulting calculation serves as a partial justification for Widom's conjecture about the homogeneity of the free energy and at the same time gives his result $sv' = \gamma' + 2\beta$.

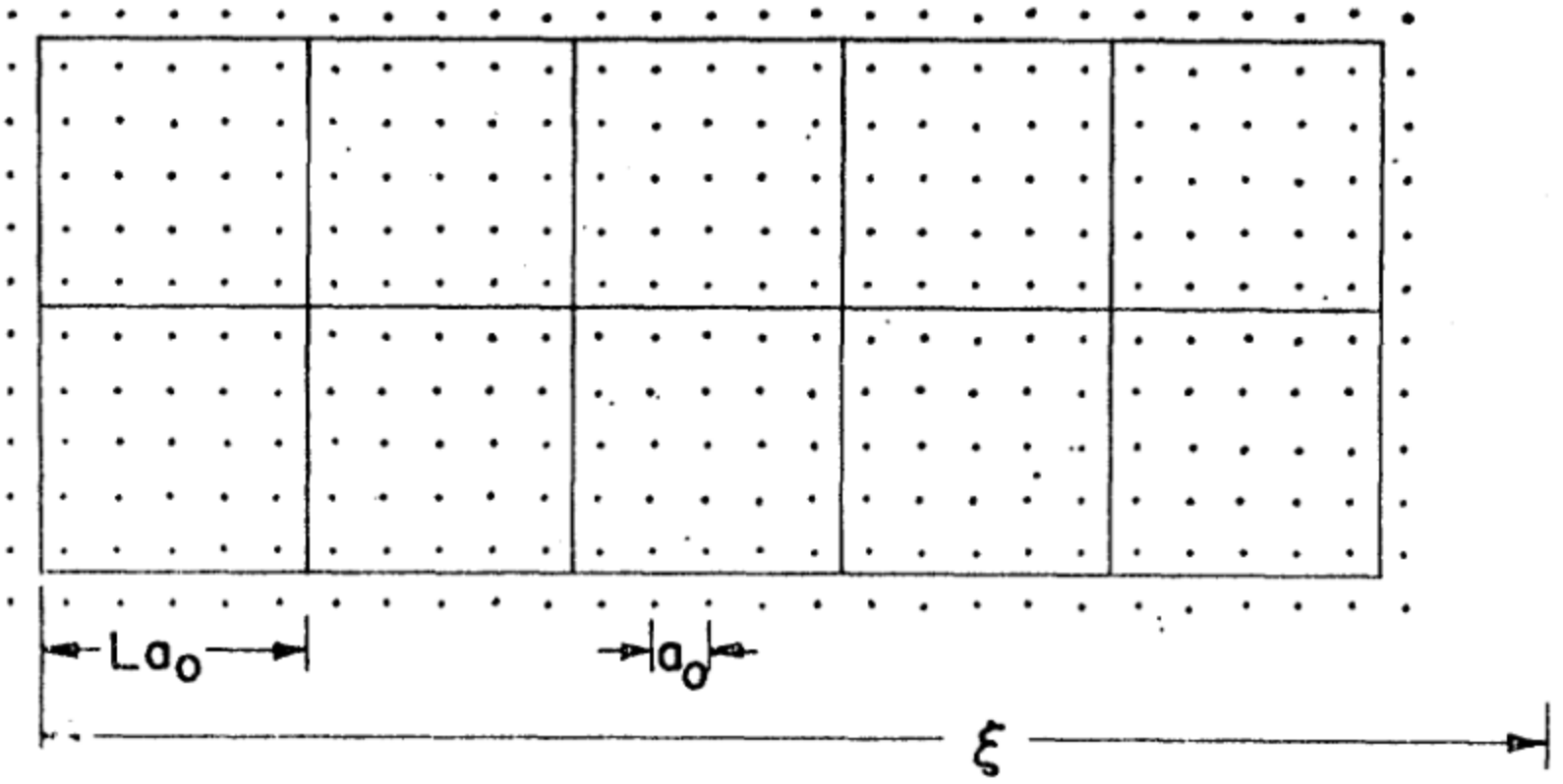

FIG. 2. Division of Ising model lattice into cells. $L \gg 1$ but $La_0 \ll \xi$.

Next, consider the interaction among cells. The free energy will tend to be larger if the spins on neighboring cells are lined up. There will tend to be a smaller contribution if they are anti-parallel. Then, in net, this tends to make a contribution to the exponential $\exp[-\beta F]$

$$\exp \sum_{\alpha,\beta} \left\{ \mu_\alpha \mu_\beta \tilde{K}(\varepsilon, L) + f_{int}(\varepsilon, L) \right\}. \tag{7}$$

Here the sum extends over nearest neighbor cells, $\tilde{K} + f_{int}$ gives the contribution to the free energy when neighboring cells are aligned and, $-\tilde{K} + f_{int}$ gives the contribution to the free energy when they are out of step. Since the direct interactions between cells which produce f_{int} occur within a distance which is very short compared to the coherence length, we assume that f_{int} is, like $f_L(\varepsilon)$, a regular function of ε, but not necessarily a regular function of L. On the other hand, $\tilde{K}$ is perhaps a somewhat more subtle beast. This describes the extra free energy that it costs to put two cells out of step. This involves, then, the rather delicate difference between the ways cells can match up when they are in step and when they are out of step. Nonetheless, it seems reasonable to assume that $\tilde{K}(\varepsilon, L)$ is also a regular function of ε; but, we assert this with somewhat less confidence than our other statements relating to this model. In writing (7) we are asserting that the correlations among cells can be totally represented by these interactions among near neighbors and that there are no less direct interactions that we need include in (7) as long ranged interactions. This statement, together with the assertion that the cell can be represented by the double valued variable, μ_α, are the two basic assumptions of this model.

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Block spin Hamiltonian is same as spin Hamiltonian

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Static Critical Phenomena

Static Phenomena Near Critical Points: Theory and Experiment

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- Significance of this early paper: first detailed confrontation of scaling theory with experiment

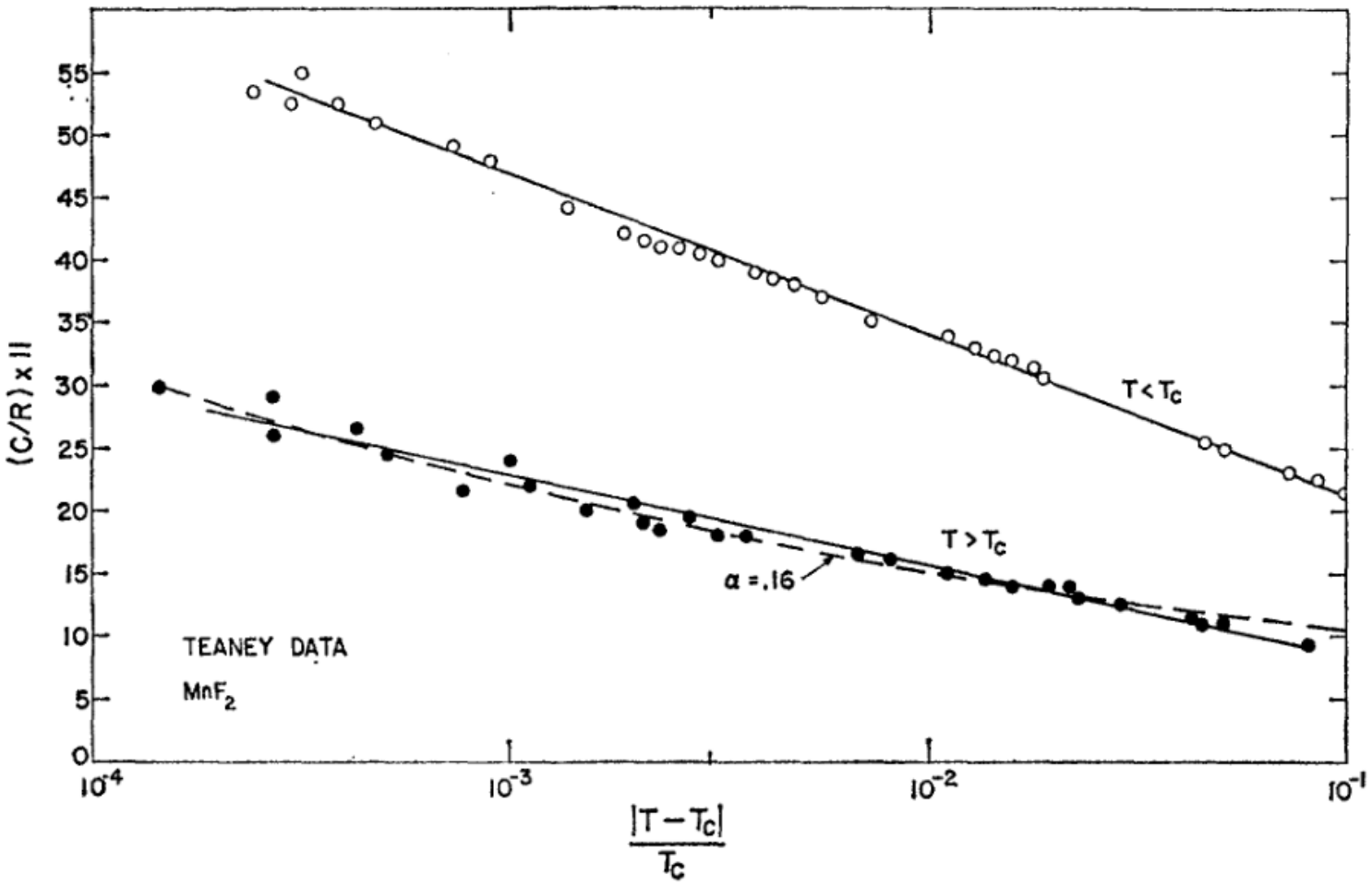
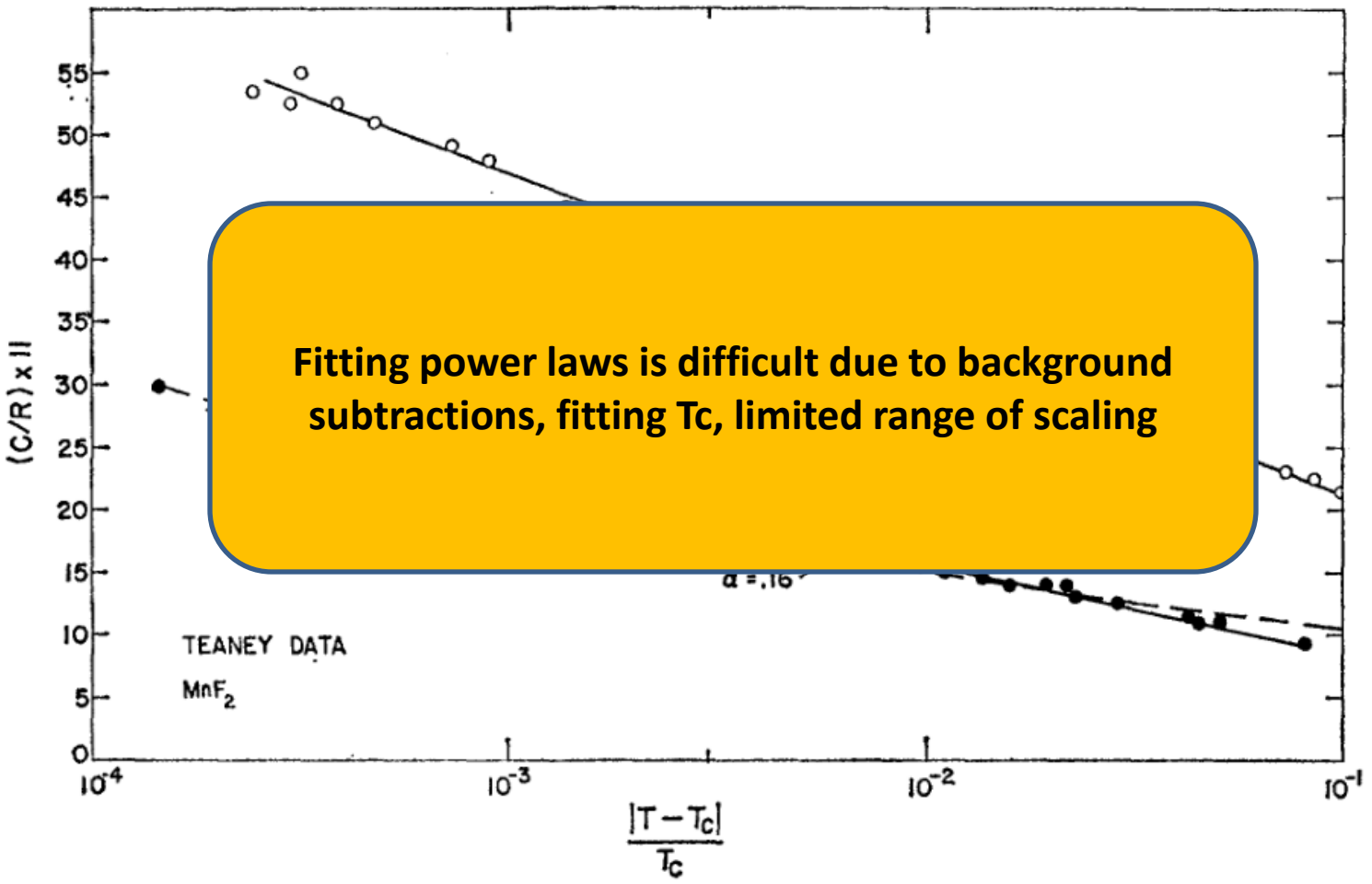


FIG. 7. Specific heat of MnF_2 as measured by Teaney. The solid lines represent logarithmic fits. The dashed line is the power law $C \sim \epsilon^{-0.16}$.



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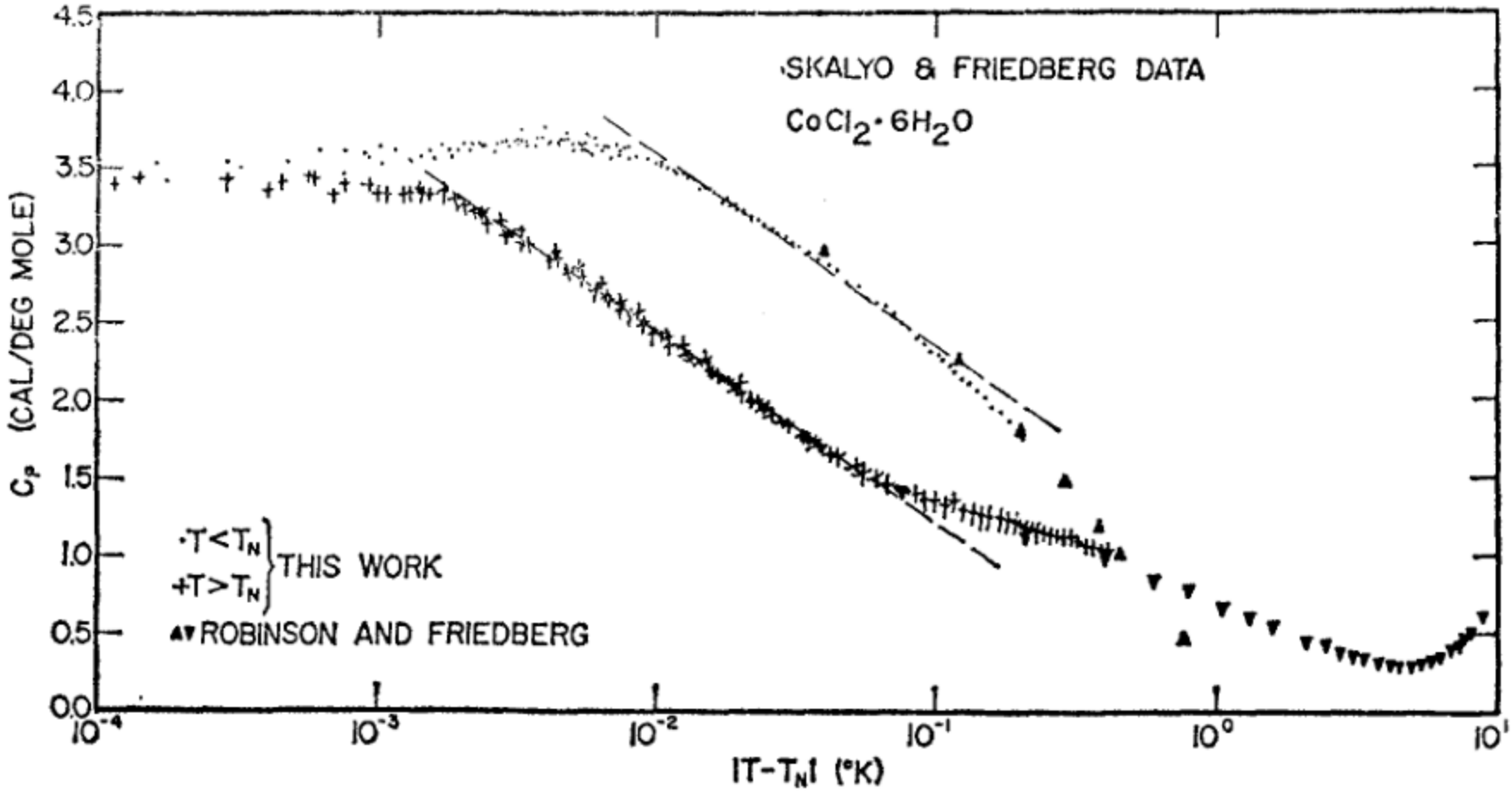

FIG. 5. Specific heat of $CoCl_2 \cdot 6\,H_2O$. The peaks round off for $\epsilon \approx 5 \times 10^{-3}$.

TABLE IX. Specific heat.[a]

Material	Experimenters	Ref.	T_c (°K)	$\epsilon = \lvert \Delta T \rvert / T_c$ Range for fit	α	α'	A'/A	Comments
				Antiferromagnets				
MnF_2	Teaney	86	67.33±0.01	2×10^{-4}–5×10^{-2}	$\lesssim$0.16	$\lesssim$0.18	1.0	
$CoCl_2\cdot6\,H_2O$	Skalyo, Friedberg	84	2.289±0.002	10^{-3}–3×10^{-2} 5×10^{-3}–4×10^{-2}	$\lesssim$0.11	$\lesssim$0.19	1.0	Rounding of peak
$MnCl_2\cdot4\,H_2O$	Friedberg, Wasscher	89	1.622±0.005	10^{-3}–10^{-1}		$\lesssim$0.14		$T>T_c$, Logarithm fits data for 4×10^{-3}–2×10^{-2}
$CuK_2(SO_4)_2\cdot6\,H_2O$	Miedema, Wielinga, Huiskamp	85	0.193±0.001	10^{-3}–2×10^{-2}	$\lesssim$0.6			
$CoCs_3Cl_5$			0.52±0.01	4×10^{-3}–2×10^{-2} 4×10^{-3}–5×10^{-2}	$\lesssim$0.7	$\lesssim$0.25		Rounding of peak
$RbMnF_3$	Teaney, Moruzzi, Argyle	90	0.83±0.01	2×10^{-4}–5×10^{-2} 2×10^{-3}–2×10^{-2}	$\lesssim$0.15	$\lesssim$0.15	1.0	Rounding of peak
				Ferromagnets				
Iron	Kraftmakher, Romashina	91	1043.0±1.0	2×10^{-3}–10^{-1} 3×10^{-3}–7×10^{-2}	$\lesssim$0.17	$\lesssim$0.13	1.0	
$CuK_2Cl_4\cdot2\,H_2O$	Miedema, Wielinga, Huiskamp	92	0.88±0.01	10^{-3}–10^{-1}	$\lesssim$0.10	$\lesssim$0.17	1.0	
Nickel	Kraftmakher	93	627.0	5×10^{-3}–8×10^{-2}				Data have considerable scatter. Logarithms with $A'/A\approx1$ provide reasonable fit.
Value used for scaling law analysis					$\lesssim$0.16	$\lesssim$0.16		
Molecular field theory					0	0		Finite discontinuity
3-dimensional Ising model					$\lesssim$0.2	$\lesssim$0.1		

[a] Further information on magnetic specific heats may be found in Ref. 1, particularly the papers of Teaney and Yamamoto *et al.*

The correct value for the specific heat exponent is about 0.11, but these measurements are difficult to do, require background subtractions, hard to get enough decades of scaling, and the critical temperature is a fitting parameter that can readily change the deduced exponents

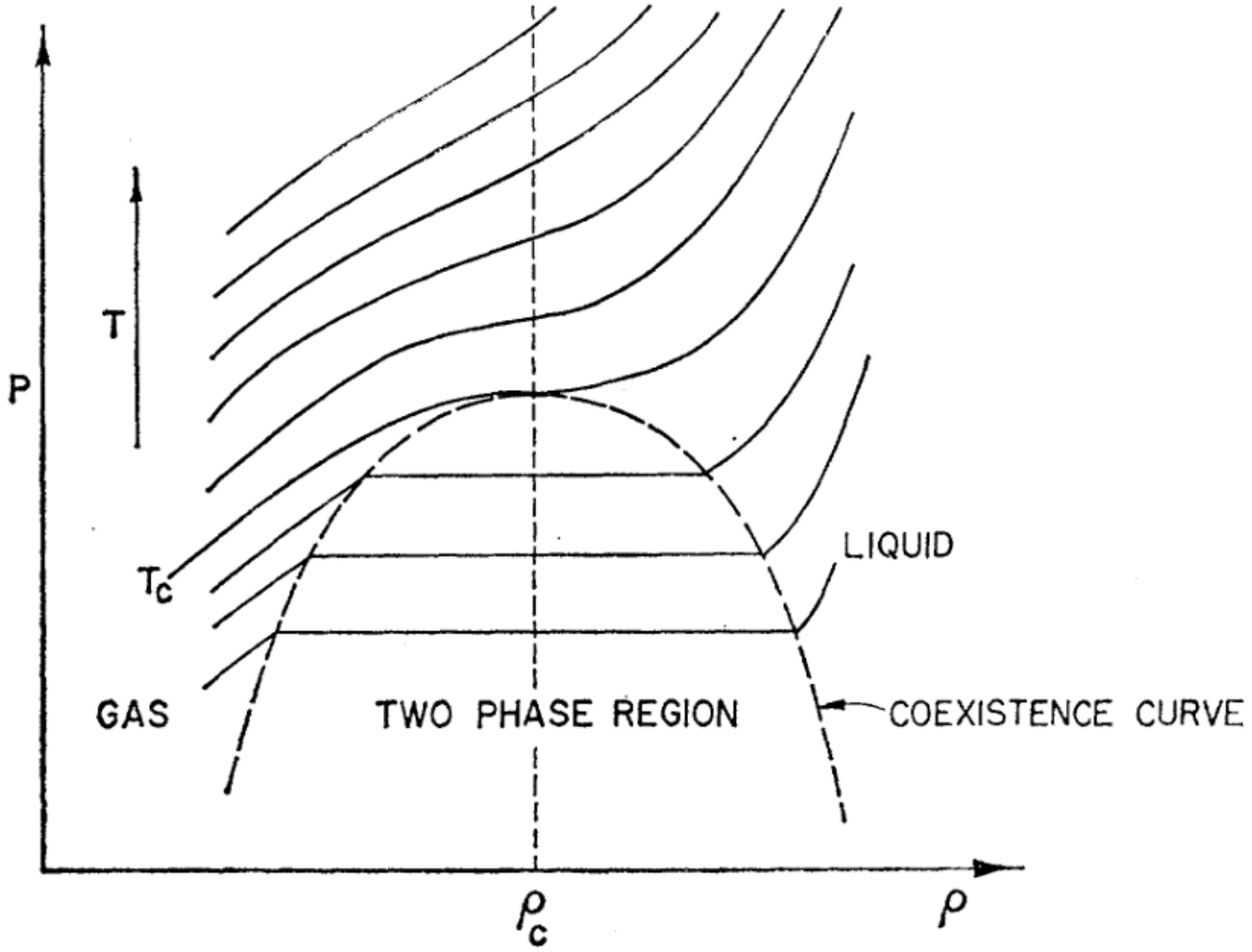

FIG. 9. Schematic pressure vs density diagram near the liquid–gas critical point.

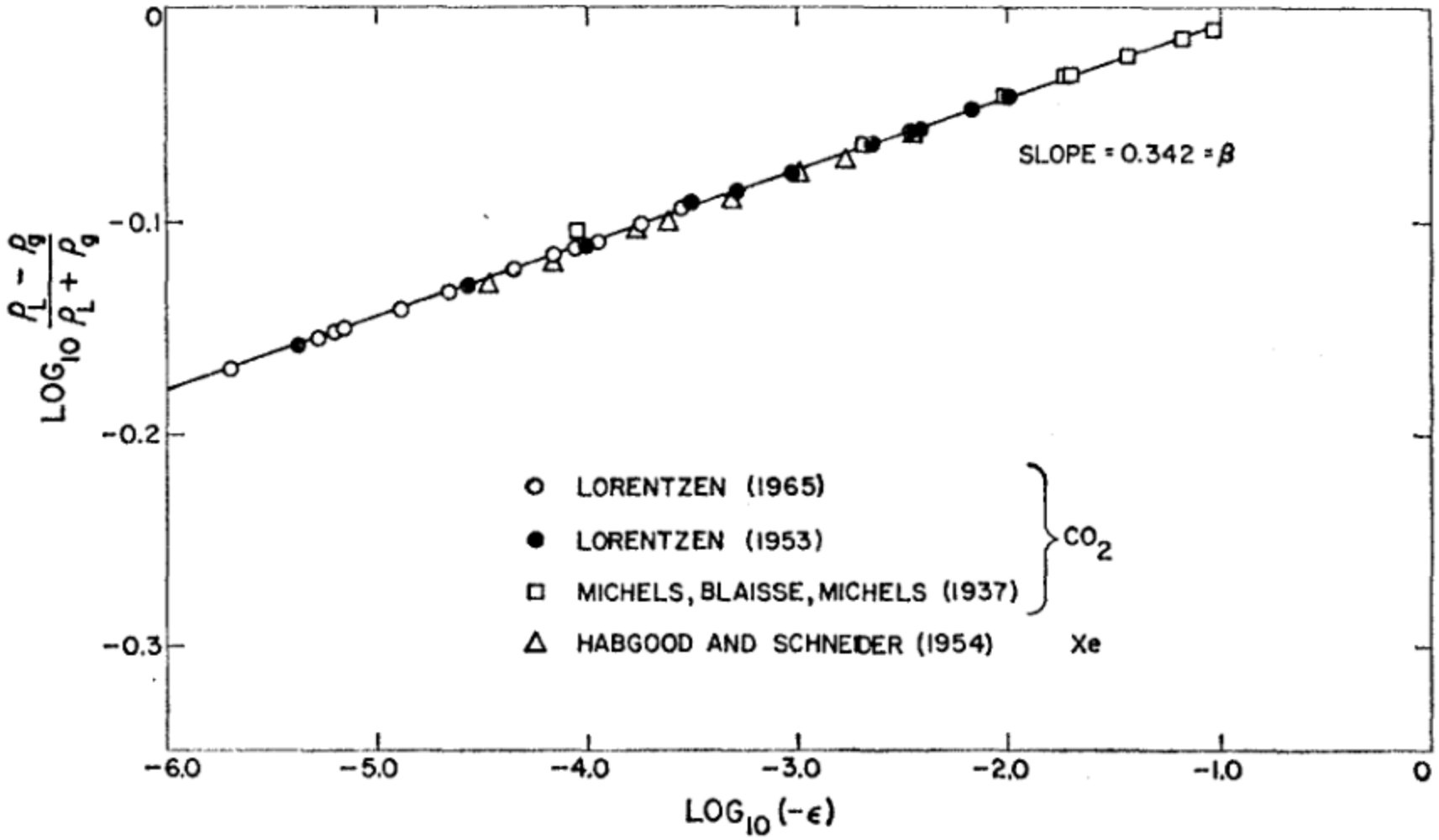

FIG. 10. Coexistence curve data for CO_2 and Xe. The critical index $\beta \approx 0.34$ over almost five decades in ϵ.

TABLE XIV. Summary table for β.

Fluid	β	Reference	Range of ϵ
Xe	0.350 ± 0.015	99	$4\times10^{-5} < -\epsilon < 4\times10^{-3}$
Ar	0.4 ± 0.2	103	$6\times10^{-5} < -\epsilon < 8\times10^{-3}$
Ar	0.33 ± 0.05	103	$8\times10^{-3} < -\epsilon < 10^{-1}$
CO_2	0.344 ± 0.01	105	$4\times10^{-6} < -\epsilon < 10^{-1}$
"Best value"	0.346 ± 0.01		
Lattice gas	0.313 ± 0.004	Table III	

Universality

TABLE XX. Comparison with scaling laws. According to the analysis of Sec. III all of these numbers should be equal. The numbers used here are the "best values" of Tables XIV–XIX.

$2-\alpha$	$2-\alpha'$	$\gamma+2\beta$	$\gamma'+2\beta$	$\beta(\delta+1)$	$\frac{3}{2}\mu$
1.8	1.88	2.06	1.7	1.87	1.91
±0.2	±0.12	±0.2	±0.3	±0.14	±0.03

Scaling laws

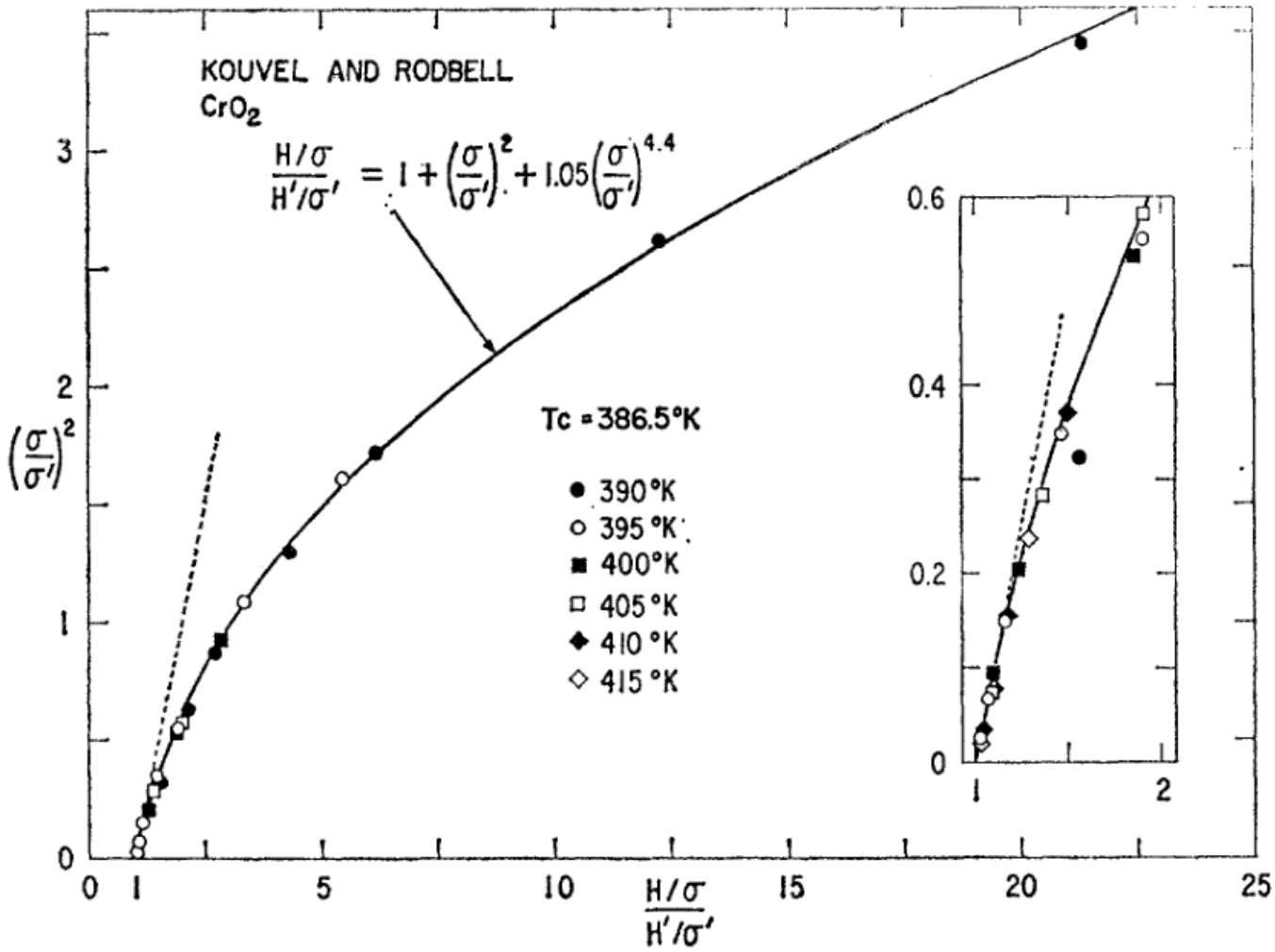

FIG. 4. Magnetization vs field for CrO_2. In our notation, $\sigma = M$, $\sigma' \sim \epsilon^{\beta}$, $H'/\sigma' \sim \epsilon^{\gamma} = \epsilon^{\beta(\delta-1)}$. Points for different ϵ fall on the same curve, which verifies the scaling law prediction, Eq. (4.1).

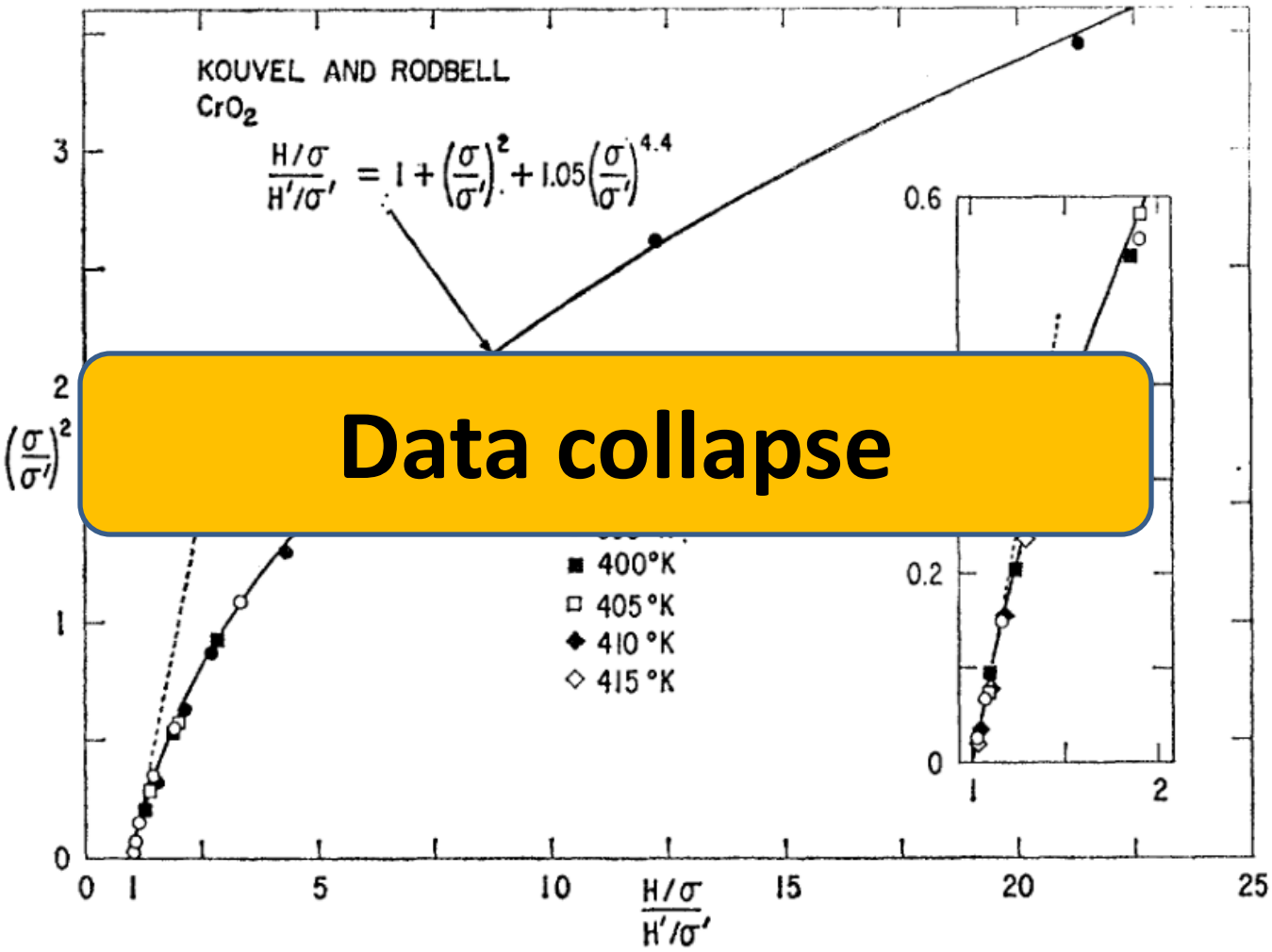

FIG. 4. Magnetization vs field for CrO_2. In our notation, $\sigma = M$, $\sigma' \sim \epsilon^{\beta}$, $H'/\sigma' \sim \epsilon^{\gamma} = \epsilon^{\beta(\delta-1)}$. Points for different ϵ fall on the same curve, which verifies the scaling law prediction, Eq. (4.1).

Kouvel[62] has used M vs H and T data to construct a very direct check of the scaling laws. According to Eq. (3.7), for a given material the magnetization obeys the relation

$$M/\epsilon^{\beta}=f(H/\epsilon^{\beta\delta}), \tag{4.1}$$

where $f(H/\epsilon^{\beta\delta})$ is an unknown function. From (4.1) we conclude that if we knew M as a function of H for one value of ϵ and if we knew β then we could predict M vs H for any other value of ϵ. [Actually this is not

These are some of the earliest tests of data collapse or Widom scaling

Static Critical Phenomena

- **Leo Kadanoff**: “The materials research lab had just been set up at Urbana, and my colleagues in the lab kindly, or maybe occasionally not-so-kindly, allowed me to commandeer their graduate students and post-docs, to give lectures in the seminar, and in the end to put together a review paper. This is something we did over a course of six or eight months. We reviewed, I believe, every experiment that we could reasonably find involving critical phenomena. And managed to fit them all into some picture which included this new scaling point of view, based, of course, on the phenomenology that had been developed by Widom. Based upon the phenomenology which had also in parallel, and earlier than my work, I believe, been developed by Patashinski and Pokrovsky. You guys could check the stuff, but although I was probably unaware of the work, the work, I believe, was previous to mine. And maybe even previous to Widom's.”

Physics of Scale Project (2004). https://authors.library.caltech.edu/5456/1/hrst.mit.edu/hrs/renormalization/public/index.html

Some (but not all) milestones in RG history

Ben Widom proposed "data collapse" (1965)

Leo Kadanoff derived data collapse from scaling concepts (1966)

Ken Wilson developed the RG based on Kadanoff's scaling ideas (1971)

PHYSICAL REVIEW B VOLUME 4, NUMBER 9 1 NOVEMBER 1971

Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture*

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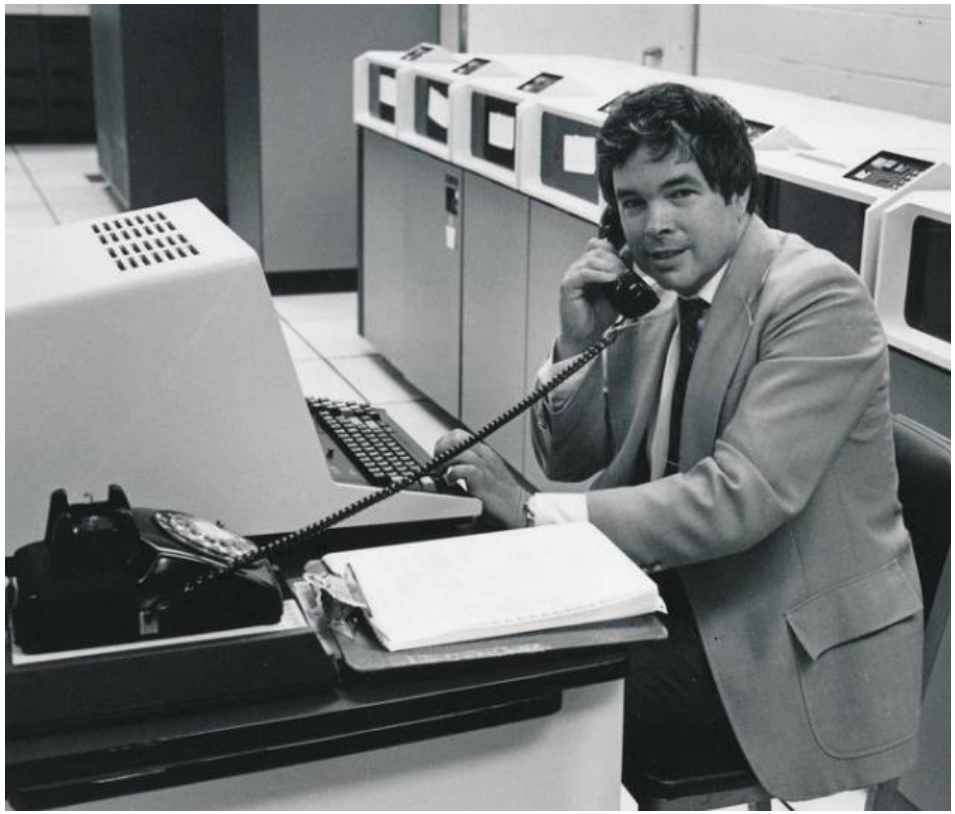
(Received 2 June 1971)

The Kadanoff theory of scaling near the critical point for an Ising ferromagnet is cast in differential form. The resulting differential equations are an example of the differential equations of the renormalization group. It is shown that the Widom-Kadanoff scaling laws arise naturally from these differential equations if the coefficients in the equations are analytic at the critical point. A generalization of the Kadanoff scaling picture involving an "irrelevant" variable is considered; in this case the scaling laws result from the renormalization-group equations only if the solution of the equations goes asymptotically to a fixed point.

RG and the Kondo Problem



- Wilson's RMP (1975)
  - Citations: 4628 (GS)

The renormalization group: Critical phenomena and the Kondo problem*†

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This review covers several topics involving renormalization group ideas. The solution of the s-wave Kondo Hamiltonian, describing a single magnetic impurity in a nonmagnetic metal, is explained in detail. See Secs. VII–IX. "Block spin" methods, applied to the two dimensional Ising model, are explained in Sec. VI. The first three sections give a relatively short review of basic renormalization group ideas, mainly in the context of critical phenomena. The relationship of the modern renormalization group to the older problems of divergences in statistical mechanics and field theory and field theoretic renormalization is discussed in Sec. IV. In Sec. V the special case of "marginal variables" is discussed in detail, along with the relationship of the modern renormalization group to its original formulation by Gell-Mann and Low and others.

CONTENTS

Introduction 773
I. Definition of a Renormalization Group Transformation 777
II. Fixed Points in Perturbation Theory 780
III. Linearized Behavior Near Fixed Points 783
IV. Divergences in Field Theory and Statistical Mechanics, the Triangle of Renormalization 787
V. Marginal Operators and Second Order Perturbations About a Fixed Point 791
VI. Block Spin Methods: The Two Dimensional Ising Model 797
VII. The Kondo Problem: Introduction and Definition of Basis 805
VIII. Renormalization Group Transformation for the Kondo Hamiltonian 816
IX. Impurity Susceptibility and Specific Heat for the Kondo Hamiltonian 825
Appendix: Perturbation Expansions for the Kondo Hamiltonian 837

INTRODUCTION

One of the most basic themes in theoretical physics is the idea that nature is described locally. The basic equations of all physics are local. For example, Maxwell's equations specify the behavior of electric and magnetic fields in an infinitesimal neighborhood of a point x. In order to be able to specify local equations it is necessary to define continuum limits, namely the limits which define derivatives. The idea of the derivative and the idea of a continuum limit that underlies the derivative is therefore of great importance in all of physics.

It is now becoming clear that there is a second form of continuum limit, called the statistical continuum limit, which also has a very broad range of applicability throughout physics. In the statistical continuum limit functions of a continuous variable are themselves *independent* variables. For example, the electric and magnetic fields throughout space can be the independent variables in a statistical continuum limit. This happens in statistical or quantum mechanical problems where there are field fluctuations, so that one has to compute averages over an ensemble of fields. In statistical calculations one does not compute the field at a point x. Instead one computes correlation functions; that is, expectation values of products of fields such as $\langle E(\mathbf{x},t)E(\mathbf{y},t')\rangle$. In quantum mechanical problems the correlation functions are sometimes replaced by vacuum expectation values of products of fields. In the simplest cases a field average determining a correlation function can be written formally as a functional integral. In the functional integral the fields are the independent variables of integration.

There are two ways in which a statistical continuum limit can arise. The obvious way is when the independent field variables are defined on a continuous space; the case of statistical or quantum fluctuations of the electromagnetic field is an example. If one were to replace the continuum by a discrete lattice of points, the field averages would consist of integrals over the value of the field E at each lattice site n. Thus for the discrete lattice case one has a multiple integration, $\prod_n \int dE_n$, the variables of integration being the fields E_n. In the continuum limit one has infinitely many integration variables E_n. Problems with infinitely many variables can be very difficult to solve.

The second source of statistical continuum limits is the situation where one has a lattice with a fixed lattice spacing, usually an atomic lattice. The number of independent variables (i.e., independent degrees of freedom) at each lattice site is fixed and finite. The continuum limit arises when one considers large size regions containing very many lattice sites. When the lattice is viewed on a macroscopic scale one normally expects the lattice structure to be invisible. That is, large scale effects should be describable by a continuum picture making no reference to the lattice spacing.

Consider, for example, critical phenomena in a magnet. A magnet is built of atoms and the atomic spacing provides a fixed shortest length which does not go to zero. At the critical point (the critical point occurs at the Curie temperature) there are long wavelength fluctuations of the magnetization signalling the onset of spontaneous magnetization. The maximum wavelength of the fluctuations is the corre-

* Supported in part by the National Science Foundation.
† This paper is a compilation of material presented as a series of nine lectures at Cargese in Summer 1973.

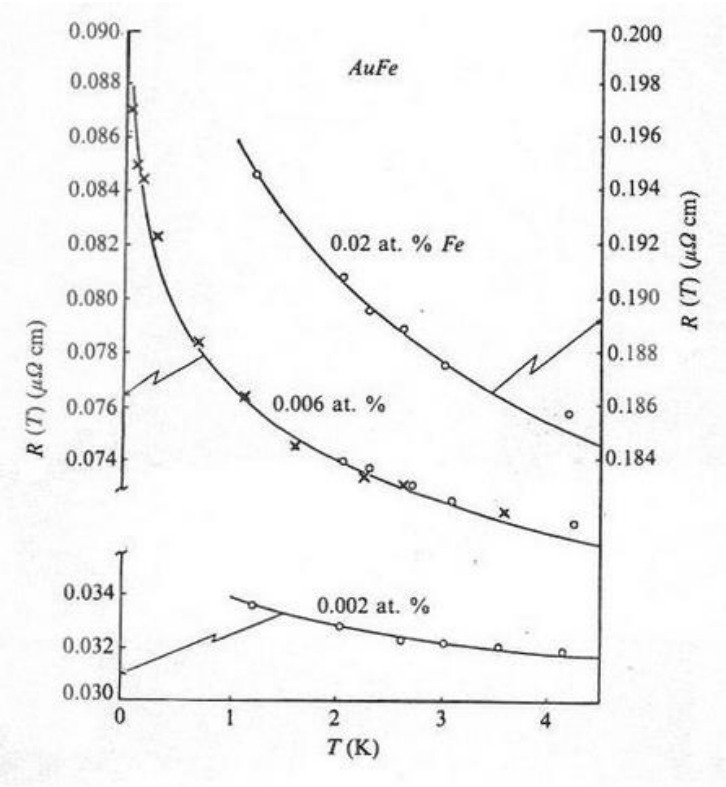
https://www.nytimes.com/2013/06/21/science/kenneth-wilson-nobel-physicist-dies-at-77.html

What is the Kondo problem?

What is the Kondo problem?

- Scattering of conduction band electrons from a single magnetic impurity
  - Resistivity
  - What is the resistivity as T → 0?

- The problem is that the calculation of the upturn leads to a divergent resistivity at T = 0.



Kondo (1964)

The state of RG c. 1975

- Wilson begins by comparing spatial derivatives and their continuum limit with functional integration …

> It is now becoming clear that there is a second form of continuum limit, called the statistical continuum limit, which also has a very broad range of applicability throughout physics. In the statistical continuum limit functions of a continuous variable are themselves *independent* variables. For example, the electric and magnetic fields throughout space can be the independent variables in a statistical continuum limit. This happens in statistical or quantum mechanical problems where there are field fluctuations, so that one has to compute averages over an ensemble of fields.

The state of RG c. 1975

A procedure is now being developed to understand the statistical continuum limit. The procedure is called the renormalization group. It is the tool that one uses to study the statistical continuum limit in the same way that the derivative is the basic procedure for studying the ordinary continuum limit. However, the problems that one studies with the renormalization group are rarely formulated explicitly in terms of continuum limits. Because of this the very general nature of the renormalization group has been less apparent than the general nature of the derivative.

The renormalization group is at a much more primitive stage than the derivative. There is only a small subset of problems involving the statistical continuum limit that have been solved so far, and to solve these problems a large amount of labor and theoretical artifice is required. One is still a long way from the simple and yet explicit nature as the derivative. Nevertheless, the renormalization group is the only method at present which is explicitly designed to investigate statistical continuum limit problems, and is likely to remain so. Also there are excellent prospects for the renormalization group to become much more powerful in the future than it is at present.

The state of RG c. 1975

Renormalization theory, due to Bethe, Schwinger, Feynman, Dyson, etc. [see Schwinger (1958)], eliminates the divergences of quantum electrodynamics. Renormalization theory was the first method developed for computing the statistical continuum limit of a local theory. It continues today to be an important tool for investigating the statistical continuum limit. However, the standard renormalization theory applies only to problems which can be solved by a Feynman diagram expansion. Even more restrictive is the requirement that only a few Feynman diagrams be important after renormalization. (There are techniques for summing infinite subclasses of Feynman diagrams but unfortunately these methods are effective only in a few cases.) The worst feature of the standard renormalization procedure is that it is a purely mathematical technique for subtracting out the divergent parts of integrals in the continuum limit. It gives no insight into the physics of the statistical continuum limit.

RG beyond Feynman diagrams

There is an analogue to universality in the case of an ordinary derivative. Namely, there are many different finite difference approximations to a single derivative. That is, many different discrete lattice differences have identical continuum limits. Universality is the corresponding result for the statistical continuum limit.

The fourth aspect of renormalization group theory is the construction of nondiagrammatic renormalization group transformations, which are then solved numerically, usually using a digital computer. This is the most exciting aspect of the renormalization group, the part of the theory that makes it possible to solve problems which are unreachable by Feynman diagrams. The Kondo problem has been solved by a nondiagrammatic computer method. The renormalization group solution of the Kondo problem is explained in detail in this paper: see Sec. VII–IX.

RG beyond Feynman diagrams

There is no other review of the renormalization group solution of the Kondo problem reported in Sec. VII–XI except an earlier cryptic report by Wilson (1974a).

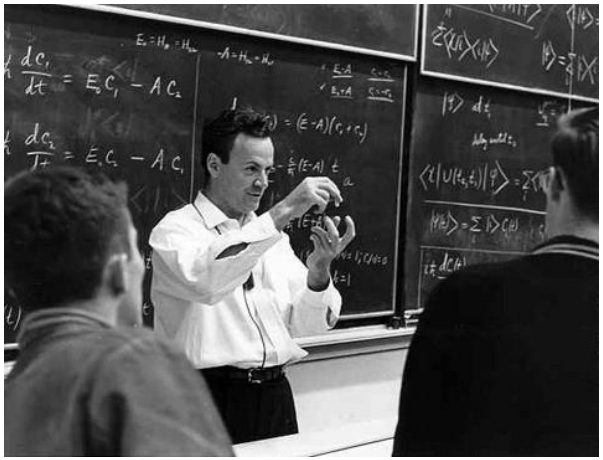
Renormalization group theory is technically more demanding than the theory of derivatives or Feynman diagrams. However, most of the unsolved problems in physics and theoretical chemistry are of the kind the renormalization group is intended to solve (other kinds of problems usually do not remain unsolved for long). It is likely that there will be a vast extension of the renormalization group over the next decade as the methods become more clever and powerful; there are very few areas in either elementary particle physics, solid state physics, or theoretical chemistry that are permanently immune to this infection.

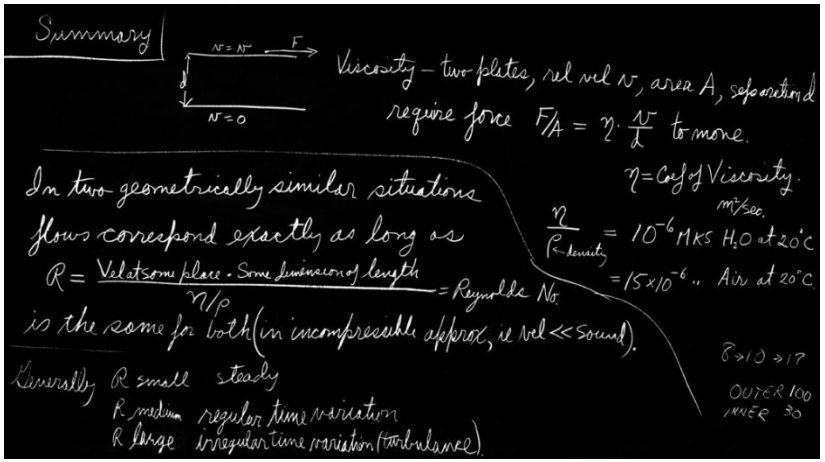
Wilson's conception of RG

The Kondo problem is an important problem in its own right. In addition, the solution of the Kondo problem is the first example where the full renormalization program (as the author conceives it) has been realized: the formal aspects of the fixed points, eigenoperators, and scaling laws will be blended with the practical aspect of numerical approximate calculations of effective interactions to give a quantitative solution (the present accuracy is a few percent) to a problem that previously had seemed hopeless. The errors of the numerical calculation have been determined (although not rigorously) as part of the calculation and can be reduced by using more computing time.

So at present the Kondo calculation sets the standards for what a renormalization group calculation can accomplish.

Wilson's conception of RG

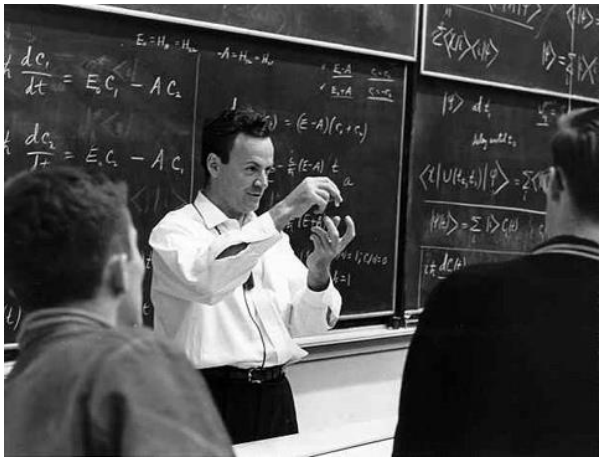


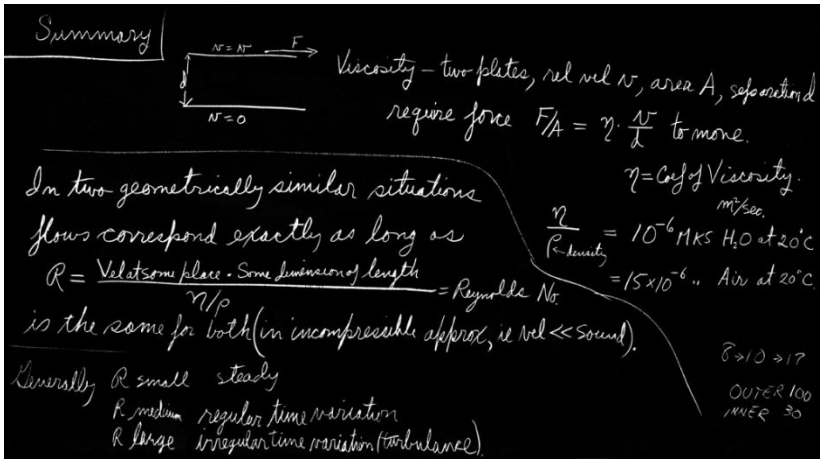


- We have written the equations of water flow. From experiment, we find a set of concepts and approximations to use to discuss the solution—vortex streets, turbulent wakes, boundary layers. When we have similar equations in a less familiar situation, and one for which we cannot yet experiment, we try to solve the equations in a primitive, halting, and confused way to try to determine what new qualitative features may come out, or what new qualitative forms are a consequence of the equations. Our equations for the sun, for example, as a ball of hydrogen gas, describe a sun without sunspots, without the rice-grain structure of the surface, without prominences, without coronas. Yet, all of these are really in the equations; we just haven't found the way to get them out.

http://www.feynmanlectures.caltech.edu/II_41.html Scientific American

Wilson's conception of RG





The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality—or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

http://www.feynmanlectures.caltech.edu/II_41.html Scientific American

PHYSICAL REVIEW E VOLUME 54, NUMBER 1 JULY 1996

Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory

Lin-Yuan Chen,[1,2] Nigel [illegible] and Y. Oono[1]
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[2]*Institute for Theoretical Physics, University [illegible]*
(Receiv[illegible]

Perturbative renormalization group theory is d[illegible] numerous examples, we illustrate its applicatio[illegible] scales, boundary layers with technically diff[illegible] conventional methods, the renormalization [illegible] structure of perturbation series nor the us[illegible] provides approximate solutions which are [illegible] latter can be reproduced, if desired, by ap[illegible] show that the renormalization group equa[illegible] of view develop reductive perturbation [illegible] systems near bifurcation points, derivin[illegible] 651X(96)00506-5]

Journal of Statistical Physics, Vol. 93, Nos. 3/4, 1998

PHYSICAL REVIEW E VOLUME 51, NUMBER 6 JUNE 1995

Numerical renormalization-group calculations for similarity solutions and traveling waves

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(Received 9 Decembe[illegible]

We present a numerical implementation of the [illegible] equations, constructing similarity solutions an[illegible] localized initial conditions, successive ite[illegible] in space and time will drive the syst[illegible] tion, such as a similarity or tr[illegible] We demonstrate that the [illegible]

PACS number(s)

CHAPTER 10

Anomalous Dimensions Far from Equilibrium

group (RG) for partial differential
how that for a large class of well-
fined discrete RG transformation
ponds to a scale-invariant solu-
ng-time asymptotic behavior.
ient.

Block Spins for Partial Differential Equations

Nigel Goldenfeld,[1] Alan McKane,[2] and Qing Hou[1]

Received February 10, 1998

We investigate the use of renormalization group methods to solve partial differential equations (PDEs) numerically. Our approach focuses on *coarse-graining* the underlying continuum process as opposed to the conventional numerical analysis method of *sampling* it. We calculate exactly the coarse grained or "perfect" Laplacian operator and investigate the numerical effectiveness of the technique on a series of 1 + 1-dimensional PDEs with varying levels of smoothness in the dynamics: the diffusion equation, the time-dependent Ginzburg–Landau equation, the Swift–Hohenberg equation, and the damped Kuramoto–Sivashinsky equation. We find that the renormalization group is superior to conventional sampling-based discretizations in representing faithfully the dynamics with a

10.1 INTRODUCTION

The solution to the problem of critical exponents given in the previous chapter may not seem very satisfying. Although we appear to have gone beyond mean field theory by the process of successively integrating out short wavelength degrees of freedom, we seem to have finessed the question: where do anomalous dimensions come from?

The present chapter addresses this question directly, in the terms of reference of chapter 7. There, we saw that anomalous dimensions reflect presence of a microscopic length scale, which affects the behaviour mic and correlation functions asymptotically close to the we also pointed out that the mathematical mechimply that, in general, a function $F(x)$ anomalous dimensions may of critical points,

Simple viscous flows: From boundary layers to the renormalization group

John Veysey II and Nigel Goldenfeld

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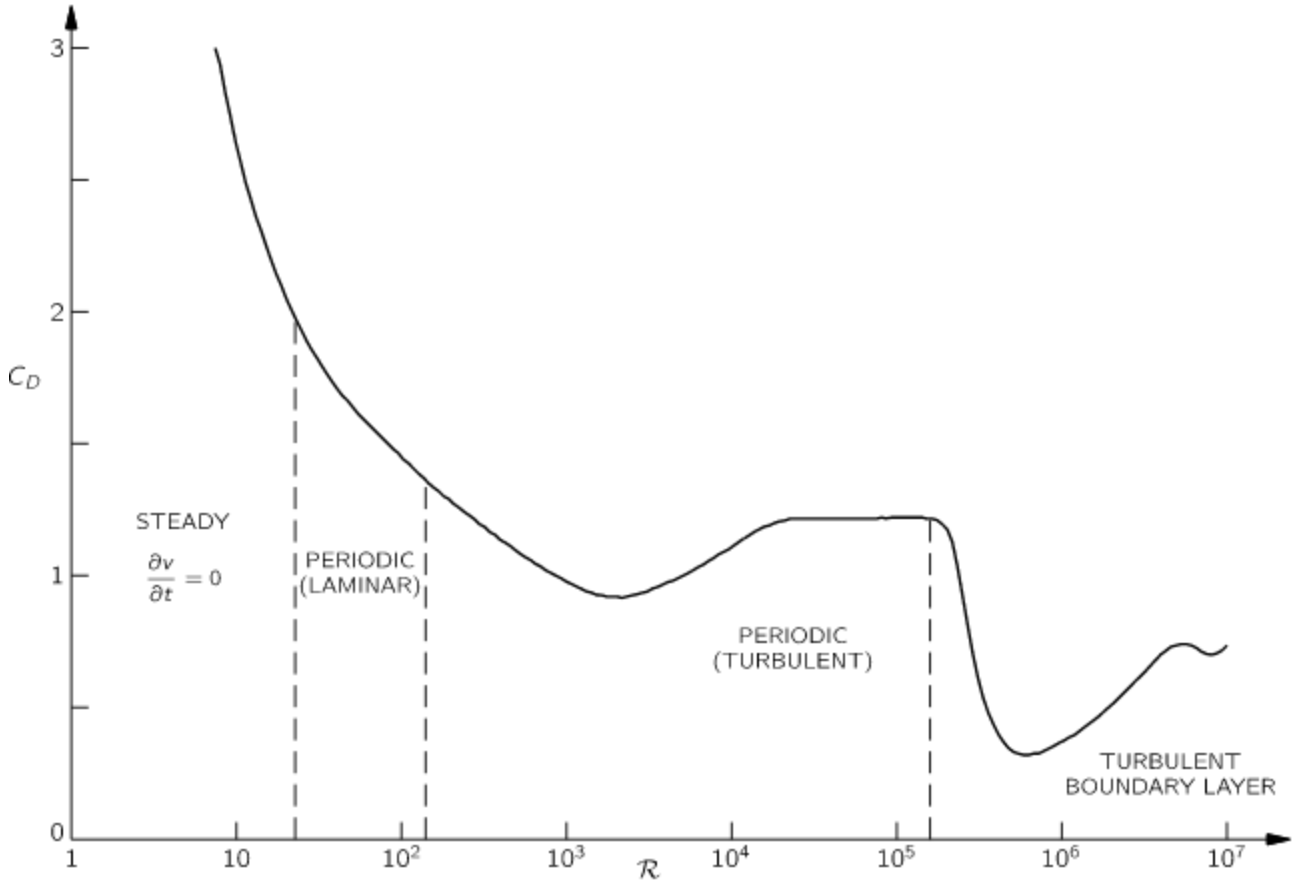
(Published 13 July 2007)



Page 34 out of 68 pages of RMP

- Up to now the article is a review of sorts, although rather idiosyncratic!
- The experience of reading it for the first time is a rewiring of one's world view about physics
- The rest is all new calculation …

RG Onions

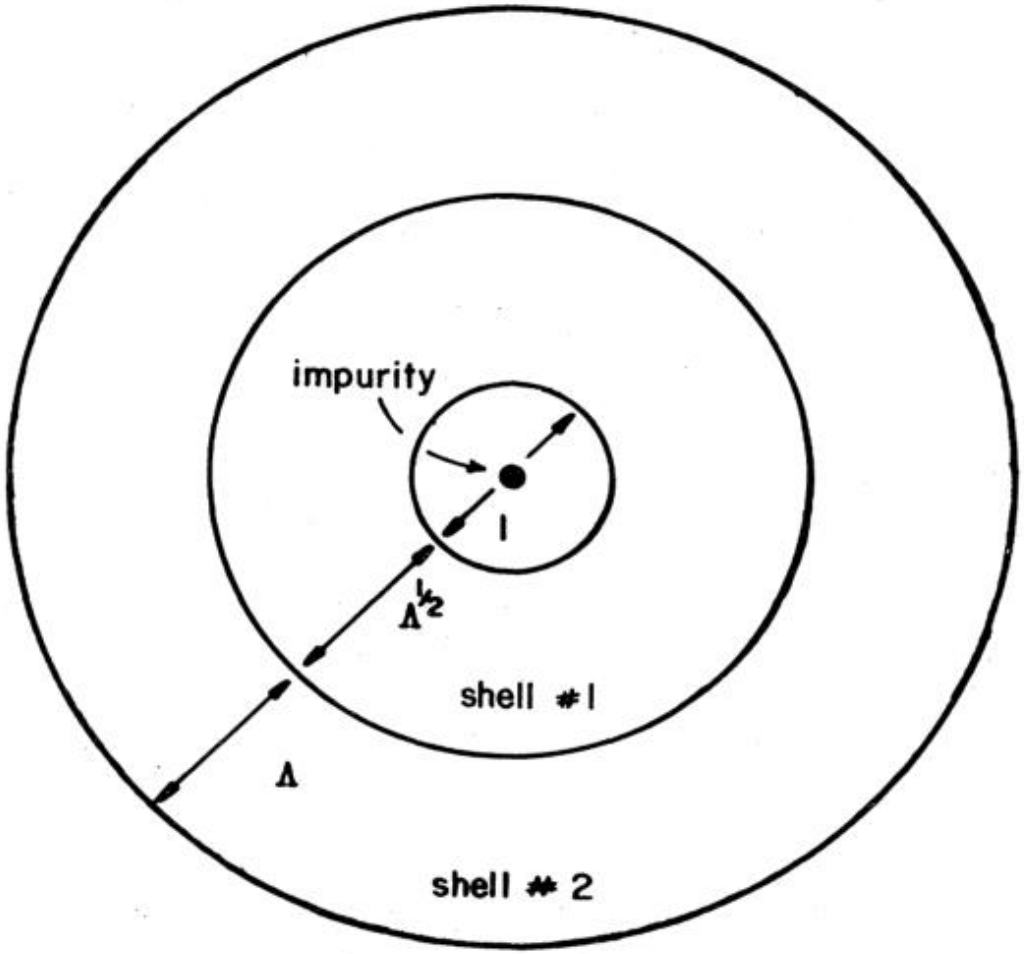


FIG. 13. Onion-like spherical shells giving the location of successive wave functions in the Kondo basis. The size of the smallest (inner) shell is a few Ångström units.

RG Railroad

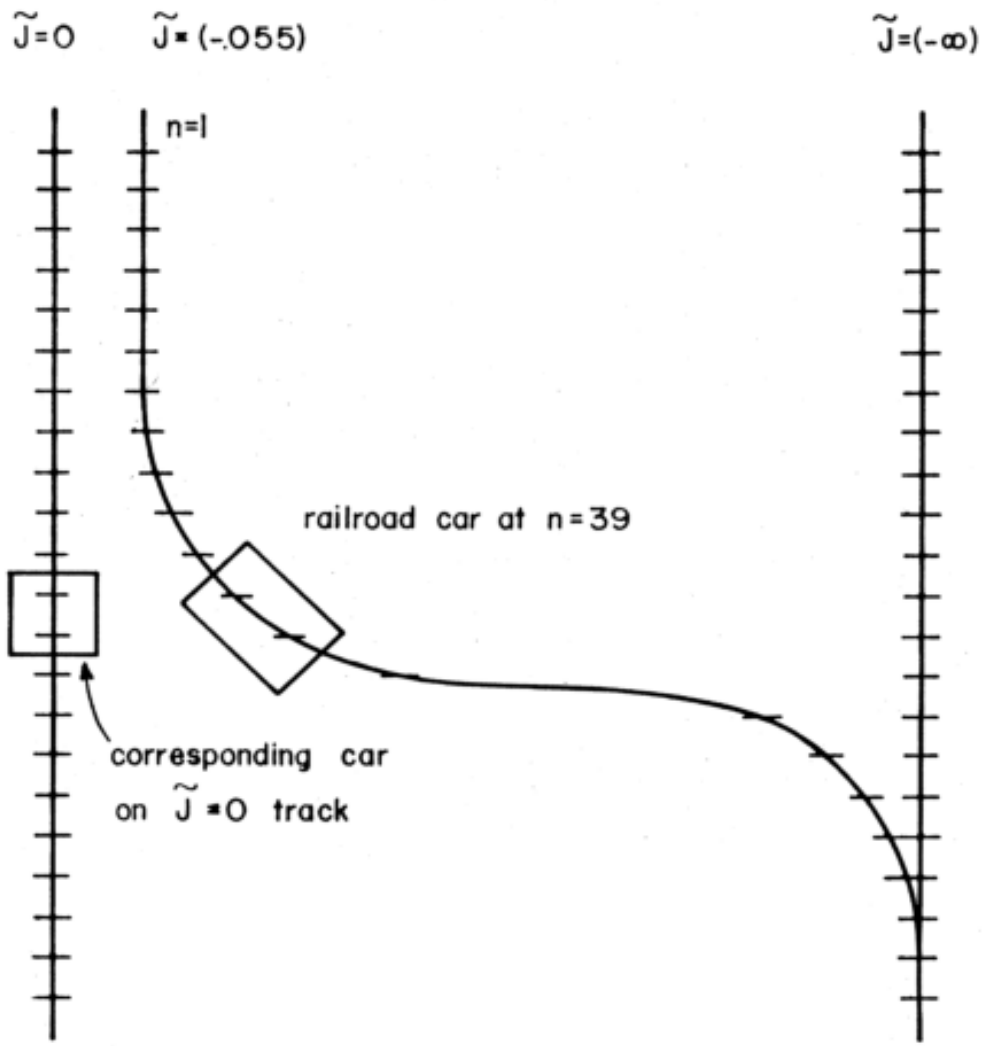


FIG. 14. Railroad track analogy for the Kondo calculation. Different tracks correspond to different initial values of $\tilde{J}$. A track from the top of the figure to the nth tie corresponds to the Kondo Hamiltonian with n electron states kept. The railroad cars illustrate the subset of energy levels actually kept in the numerical calculations.

TABLE VII. Comparison of numerical calculations with analytic predictions from the strong coupling analysis.

State	Q	S	$E(H^*)$	$E(H_N) - E(H^*)$, $N = 132 \times 10^{-4}$: Computed	Theory	$E(H_N) - E(H^*)$, $N = 144 \times 10^{-4}$: Computed	Theo
$\xi_1^*\|0\rangle$	1	½	0.6555132	− 7.015	input	−1.101	inp
$\xi_2^*\|0\rangle$	1	½	1.9760066	−21.833	−21.826*	−3.791	−3
$\xi_1^*\xi_1^*k_1^*\|0\rangle$	1	½	1.9665404	−23.384	−23.380*	−3.307	−3
$\xi_1^*\xi_1^*\|0\rangle$	2	0	1.31102801	− 0.185	− 0.181	−0.031	−(
$\xi_1^*\xi_2^*\|0\rangle$	2	0	2.6315262	0.459	0.468	0.065	
$\xi_1^*\xi_2^*\|0\rangle$	2	1	2.6315174	−31.160	−31.175	−4.892	
$\xi_2^*\xi_2^*\|0\rangle$	2	0	1.3110274	− 0.181	input	−0.034	
$\xi_2^*k_1^*\|0\rangle$	0	0	1.3110256	−27.884	−27.880	−4.370	
$\xi_2^*k_1^*\|0\rangle$	0	1		− 7.015 × 10^{-4} × 2^{66}		−1.101 × 10^{-4} ×	
λ^*				13.850 × 10^{-4} × 2^{66}		2.1687 × 10^{-4}	
w^*							

To test Eq. (VIII.38) for the difference $H_N - H^*$, the difference $H_N - H^*$ (H^* given by iteration $N = 180$) was calculated from the numerical results for two values of N: $N = 132$ and $N = 144$. The results were compared with the theoretical formulae of the table, with λ^* and w^* determined independently for each N. Two of the eight eigenvalues in Table VI were used as input to determine λ^* and w^*; the remaining 6 eigenvalues could then be predicted from the formulae of the Table. The results are shown in Table VII:

The numbers marked with an asterisk in Table VII had to be computed specially by degenerate perturbation theory. The two states involved have the same quantum numbers and for $N = 132$ the perturbation due to 0_1 and 0_3 is about the same order as the difference of the unperturbed energies. So it was necessary to take into account the off diagonal matrix element of the perturbation and diagonalize the two by two matrix from (H^* + perturbation) to obtain the correct theoretical predictions. The off diagonal matrix element is 1.069 w^*. The agreement of the computer results to theoretical predictions is excellent. The values of λ^* and w^* for $N = 132$ and $N = 144$ agree to within 0.5%. This means the eigenvalues of the linearized numerical transformation have to be within 0.1% of 2. One concludes that the theoretical analysis leading to Eq. (VIII.38) is correct and complete and that the coefficients $w_1(\bar{J})$ and $w_3(\bar{J})$ can be determined to about 0.5% accuracy. [Eqs. (VIII.39) and (VIII.40) relate w_1 and w_3 to λ^* and w^*.]

One final point for large N. The constants w_1 and w_3 have been determined for an example of small negative $\bar{J}$, namely $\bar{J} = -0.024$. They can also be determined for $\bar{J} = -\infty$. Even for $\bar{J} = -\infty$, the interactions H_N for finite N are *not* equal to H^* and for large N the form (VIII.38) should still be valid. For $\bar{J} = -\infty$, H_N is a quadratic form in the g's and k's, so the nonquadratic term $(f_0^*f_0 - 1)^2$ cannot occur in the linearized approximation; hence $w_3(J)$ (i.e., w^*) must vanish. The value of w_1 (i.e., λ^*) can be determined from the N de the single particle energies of H_N. From num nalizations one finds the N dependence ($\Lambda =$ to H_N with $\bar{J} = -\infty$) shown in Table VI $\lambda_* \simeq 0.44$.

This concludes the analysis of the nume N very large. The next topic is the analysi small values of N, where the $\bar{J} = 0$ fixed (assuming the initial value of $\bar{J}$ is small discussion will be given. Near the $\bar{J} = 0$ is one marginal eigenoperator; the rest a the largest irrelevant eigenvalue being general discussion of marginal operator one expects the following. For sufficie enough so that the irrelevant operator H_N will be completely determined ex eter (the parameter g_t of Sec. V). dently of the original value of $\bar{J}$. Thi in Sec. V, that there will be discret values $\bar{J}$ which lead to the same actions H_N, except for a translatio has been confirmed numerically. ferent initial values of $\bar{J}$, for Λ for the first and second excited st shown in Table IX.

The second run had a smaller Starting with iteration 20 there ing. For small N, irrelevant op the two runs diverge from ea for $N = 20$ apply to every calculations, not just the exan

TABLE VIII. Single particle energies for $\bar{J} = -\infty$, vs N.

	$N = 30$	$N = 32$	$N = \infty$	$(N = 30) - (N = \infty)$
w_1	0.65552674	0.65551984	0.65551294	1.38 × 10^{-5}
w_3	1.97605015	1.97602630	1.97600245	2.385 × 10^{-5}
v_3	4.00006262	3.99997201	3.99988140	9.061 × 10^{-5}

TABLE IX. Verification (from n is equivalent to a change in N wh show two excited state energies three columns are for $\bar{J} = -0.0$

N	$\bar{J} = -0.055016$ E_1	E_2
6	0.075257	0.1006
8	0.07777	0.104
10	0.08156	0.109
20	0.11367	0.15
30	0.18440	0.27
40	0.38734	0.5
50	0.68168	1.3

TABLE XIV. Values of E, F, and G in Eq. (IX.58).

Λ	2.0	2.25	2.5	3
E	$-0.5 \pm 0.004\%$	$-0.5 \pm 0.04\%$	$-0.5 \pm 0.2\%$	$-0.5 \pm 1\%$
F	-0.5	-0.5	-0.5	-0.5
G	$3.16485 \pm 0.01\%$	$3.1648 \pm 0.1\%$	$3.165 \pm 0.3\%$	$3.165 \pm 1.5\%$

The expansions for z_N and z_{N+2L} can be substituted in Eq. (IX. 44). Writing $y_L = 2kT\chi(T) - 0.5$, the result is

$$\Phi(y_L) - \Phi(y_0) = -L \ln 2 - \ln(T/T_0), \tag{IX.47}$$

where

$$\Phi(y_0) = (\ln 2)\Psi(z_N) + 1.0018 = (-1/2y_0) - 0.5 \ln|2y_0| + 3.1648y_0 + O(y_0^2). \tag{IX.48}$$

The constant 1.0018 was added in Eq. (IX.48) so that $\Phi(y_0)$ would have no constant term. Now one has

$$\Phi[2kT\chi(T) - 0.5] - \Phi[2kT_0\chi(T_0) - 0.5] = \ln(T/T_0) \tag{IX.49}$$

which means that if one defines B to be

$$B = \ln kT - \Phi[2kT\chi(T) - 0.5], \tag{IX.50}$$

then B is independent of T so long as T is an inverse power of 2.

The quantity B depends on the initial coupling constant J. In order to make connections with previous work on the Kondo problem we shall return to the constant J of previous workers in place of $\bar{J}$. In the conventional continuum Kondo theory discussed in (Kondo, 1969), $\chi(T)$ has an expansion of the form

$$kT\chi(T) = 0.25\{1 + 2J_\rho + 4J^2\rho^2 \ln(kT/D) + \cdots\}, \tag{IX.51}$$

where ρ is the density of states at the Fermi surface, and D is the width (in energy) of the conduction band. (In the model of Sec. VII, ρ is 2, and D is 1.) Only the leading logarithm has been calculated in order $J^2\rho^2$.

Consider B as a function of J_ρ. It is convenient to write B in terms of a new constant $\bar{D}$

$$B(J_\rho) = \ln\bar{D}(J_\rho) - \Phi(J_\rho). \tag{IX.52}$$

The point of this equation is that $\bar{D}(J_\rho)$ has a power series expansion in J_ρ while $B(J_\rho)$ does not. This follows from the fact that $\chi(T_0)$ for fixed T_0 has a power series expansion in J_ρ, and in particular $y_0 = 2kT\chi(T_0) - 0.5$ has the form

$$y_0 = J_\rho + O(J^2\rho^2). \tag{IX.53}$$

From Eq. (IX.50) one has

$$\ln[kT_0/\bar{D}(J_\rho)] = \Phi[y_0] - \Phi[J_\rho]. \tag{IX.54}$$

The difference $\Phi(y_0) - \Phi(J_\rho)$ contains the differences $[y_0^{-1} - (J_\rho)^{-1}]$ and $\ln y_0 - \ln(J_\rho)$. Using Eq. (IX.53) it follows that each of these differences has a power series expansion in J_ρ. Using the second order term in Eq. (IX.51), one obtains

$$\bar{D}(J_\rho) = D(c_0 + c_1J_\rho + c_2J^2\rho^2 + \cdots). \tag{IX.55}$$

The constants c_0, c_1, etc. are determined by the constant (nonlogarithmic) terms in the expansion (IX.51) in order $J^2\rho^2$, $J^3\rho^3$, etc. An example will be given later.

The temperature dependence of $\chi(T)$ for $T \gg T_K$ is now given by

$$\Phi[2kT\chi(T) - 0.5] = \Phi[J_\rho] + \ln[kT/\bar{D}(J_\rho)]. \tag{IX.56}$$

Solving this equation as an expansion in J_ρ, using Eq. (IX.48), one obtains

$$kT\chi(T) = 0.25\{1 + 2J_\rho + 4J^2\rho^2 \ln(kT/\bar{D}) + J^2\rho^2[8 \ln^2(kT/\bar{D}) + 4 \ln(kT/\bar{D})] + J^3\rho^3[16 \ln^3(kT/\bar{D}) + 20 \ln^2(kT/\bar{D}) - 21.33 \ln(kT/\bar{D})] + \cdots\}. \tag{IX.57}$$

To the author's knowledge the nonleading logarithms in this expansion have not been previously calculated to this order in J_ρ. The leading logarithms agree with earlier calculations.

The function $\Phi(y)$ in principle should be calculated in the limit $\Lambda \to 1$. We need to know the errors that occur for $\Lambda \geq 2$. The general form of $\Phi(y)$ is

$$\Phi(y) = (E/y) + F \ln|2y| + Gy, \tag{IX.58}$$

where E, F, and G depend on Λ; except for F they also depend on the value of $\bar{\beta}$ chosen in calculating the expansion of χ in terms of z_N. These constants have been calculated for a range of values of $\bar{\beta}$, with the result shown in Table XIV.

Once again there is rapid convergence as Λ decreases; this will be explained later.

The equations for the temperature dependence of $\chi(T)$ obtained above were derived assuming $kT = \Lambda^{-(N-1)/2}/\bar{\beta}$ for integer N and fixed $\bar{\beta}$; they are not (as derived) valid for all T. To cover a continuous range of T one must relate the calculations of $\chi(T)$ for a continuous range of $\bar{\beta}$. This can be done in practice as follows. For two values of $\bar{\beta}$, say $\bar{\beta}$ and $\bar{\beta}'$, one can calculate $\chi(T)$ for fixed N. Thus one calculates $\chi(T)$ for $kT = \Lambda^{-(N-1)/2}/\bar{\beta}$ and $\chi(T')$ with $kT' = \Lambda^{-(N-1)/2}/\bar{\beta}'$. The result can be written

$$\Phi[2kT\chi(T) - 0.5] = (\ln\Lambda)\Psi(z_N) - H(\bar{\beta}), \tag{IX.59}$$

$$\Phi[2kT'\chi(T') - 0.5] = (\ln\Lambda)\Psi(z_N) - H(\bar{\beta}'), \tag{IX.60}$$

where $H(\bar{\beta})$ is a constant for given $\bar{\beta}$. In these formulae the small variations of Φ with $\bar{\beta}$ have been ignored. Numerical calculations of $H(\bar{\beta})$ show that to a good approximation

$$H(\bar{\beta}) = +\ln\bar{\beta} + H, \tag{IX.61}$$

where H is independent of $\bar{\beta}$. The results for H are:

Λ	2	2.25	2.5	3
H	-0.65528 $\pm 0.03\%$	-0.6424 $\pm 0.2\%$	-0.628 $\pm 0.8\%$	-0.596 $\pm 2\%$

Wilson's result

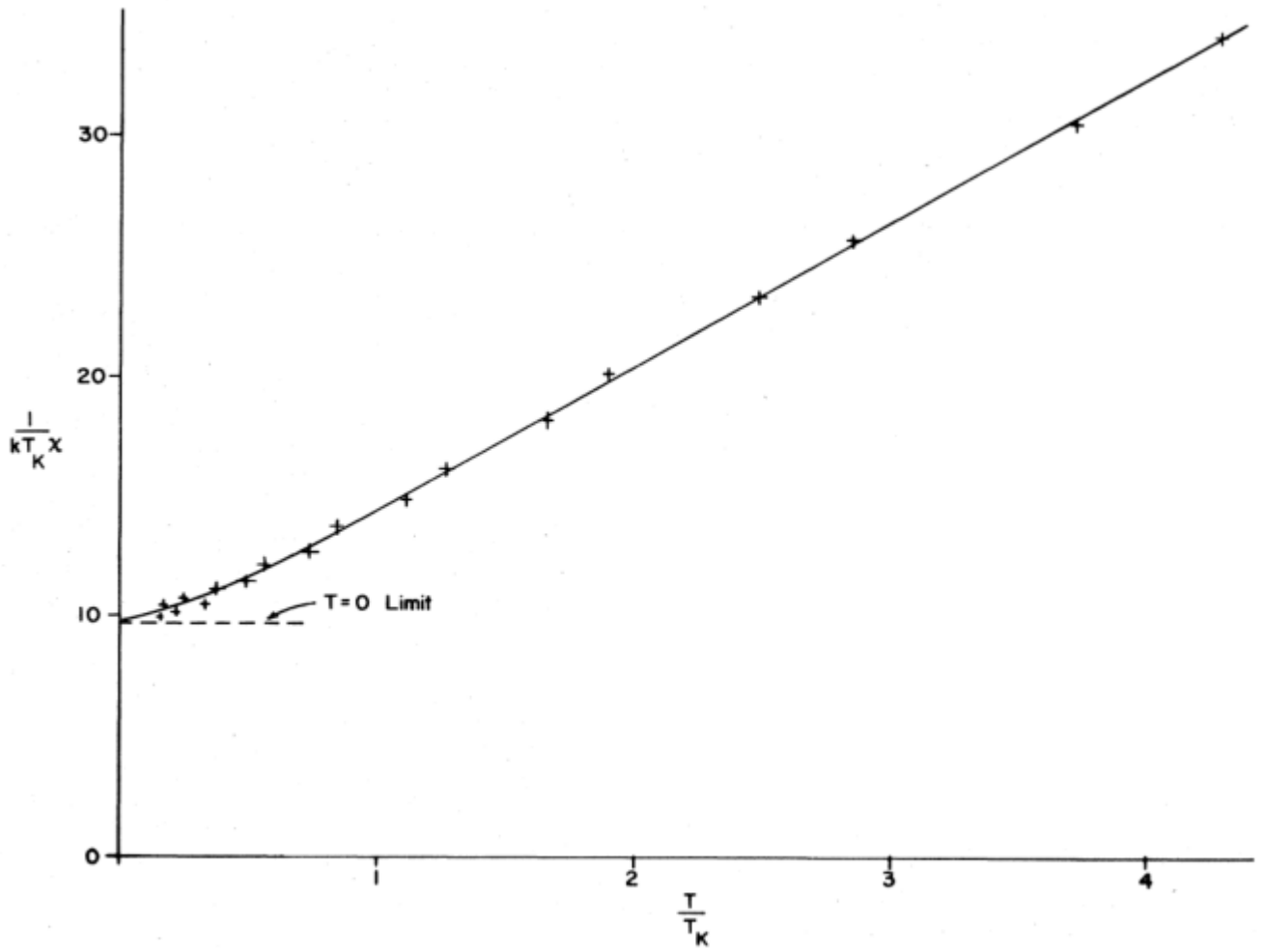
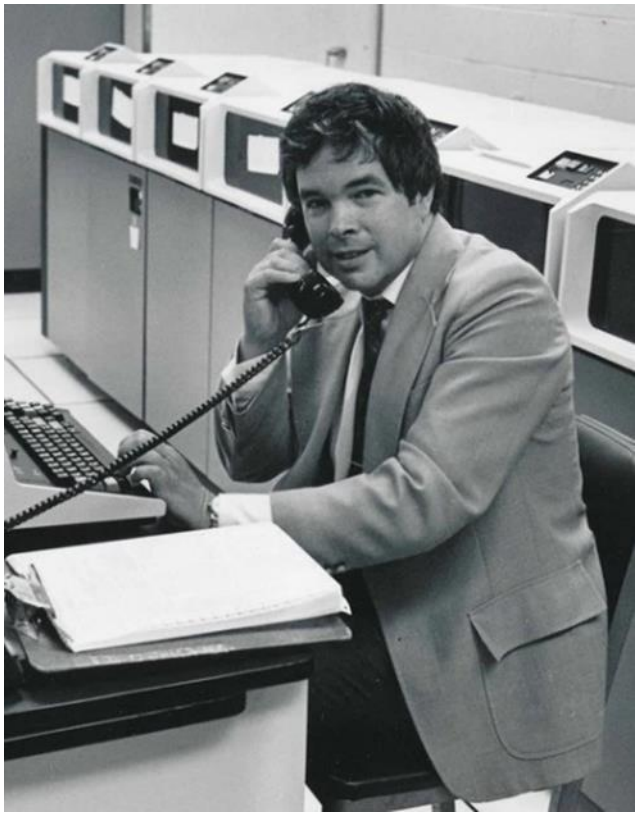


FIG. 17. Plot of inverse susceptibility $\chi^{-1}(T)$ vs T from the computer calculations. The magnetic moment and g factor of both the impurity and the conduction band electrons are set equal to 1. The plot actually shows $(kT_K\chi)^{-1}$ vs T/T_K, where T_K was defined in Eq. (IX.94). The crosses represent results for two different calculations (both with $\Lambda = 2.25$, but $\bar{I} = 0.024$ for one and 0.02412 for the other); the scatter at small T is due to truncation errors. The zero temperature value is taken from Eq. (IX.69) and Table XVI.

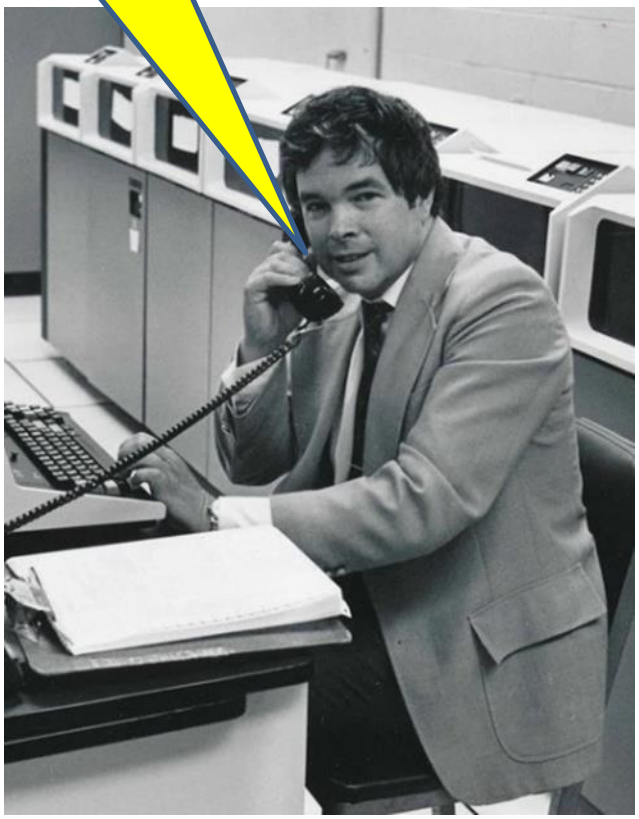
What is the Kondo problem?



What is the Kondo problem?

What is the Kondo problem?

I guess fluctuations at intermediate wavenumbers don't spark joy



Wilson's RG solution of the Kondo problem was the first example of a non-perturbative and controlled calculation of a strong coupling problem and established the RG as a powerful method beyond Feynman diagrams

Statistical Mechanics of Cellular Automata

- Wolfram's RMP (1983)
  - Citations: 3419 (GS)

Statistical mechanics of cellular automata

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Cellular automata are used as simple mathematical models to investigate self-organization in statistical mechanics. A detailed analysis is given of "elementary" cellular automata consisting of a sequence of sites with values 0 or 1 on a line, with each site evolving deterministically in discrete time steps according to definite rules involving the values of its nearest neighbors. With simple initial configurations, the cellular automata either tend to homogeneous states, or generate self-similar patterns with fractal dimensions $\simeq 1.59$ or $\simeq 1.69$. With "random" initial configurations, the irreversible character of the cellular automaton evolution leads to several self-organization phenomena. Statistical properties of the structures generated are found to lie in two universality classes, independent of the details of the initial state or the cellular automaton rules. More complicated cellular automata are briefly considered, and connections with dynamical systems theory and the formal theory of computation are discussed.

CONTENTS

I. Introduction 601
II. Introduction to Cellular Automata 602
III. Local Properties of Elementary Cellular Automata 607
IV. Global Properties of Elementary Cellular Automata 621
V. Extensions 630
VI. Discussion 639
Acknowledgments 641
References 642

I. INTRODUCTION

The second law of thermodynamics implies that isolated microscopically reversible physical systems tend with time to states of maximal entropy and maximal "disorder." However, "dissipative" systems involving microscopic irreversibility, or those open to interactions with their environment, may evolve from "disordered" to more "ordered" states. The states attained often exhibit a complicated structure. Examples are outlines of snowflakes, patterns of flow in turbulent fluids, and biological systems. The purpose of this paper is to begin the investigation of cellular automata (introduced in Sec. II) as a class of mathematical models for such behavior. Cellular automata are sufficiently simple to allow detailed mathematical analysis, yet sufficiently complex to exhibit a wide variety of complicated phenomena. Cellular automata are also of sufficient generality to provide simple models for a very wide variety of physical, chemical, biological, and other systems. The ultimate goal is to abstract from a study of cellular automata general features of "self-organizing" behavior and perhaps to devise universal laws analogous to the laws of thermodynamics. This paper concentrates on the mathematical features of the simplest cellular automata, leaving for future study more complicated cellular automata and details of applications to specific systems. The paper is largely intended as an original contribution, rather than a review. It is presented in this journal in the hope that it may thereby reach a wider audience than would otherwise be possible. An outline of some of its results is given in Wolfram (1982a).

Investigations of simple "self-organization" phenomena in physical and chemical systems (Turing, 1952; Haken, 1975, 1978, 1979, 1981; Nicolis and Prigogine, 1977; Landauer, 1979; Prigogine, 1980; Nicolis *et al.*, 1981) have often been based on the Boltzmann transport differential equations (e.g., Lifshitz and Pitaevskii, 1981) (or its analogs) for the time development of macroscopic quantities. The equations are obtained by averaging over an ensemble of microscopic states and assuming that successive collisions between molecules are statistically uncorrelated. For closed systems (with reversible or at least unitary microscopic interactions) the equations lead to Boltzmann's H theorem, which implies monotonic evolution towards the macroscopic state of maximum entropy. The equations also imply that weakly dissipative systems (such as fluids with small temperature gradients imposed) should tend to the unique condition of minimum entropy production. However, in strongly dissipative systems, several final states may be possible, corresponding to the various solutions of the polynomial equations obtained from the large time limit of the Boltzmann equations. Details or "fluctuations" in the initial state determine which of several possible final states are attained, just as in a system with multiple coexisting phases. Continuous changes in parameters such as external concentrations or temperature gradients may lead to discontinuous changes in the final states when the number of real roots in the polynomial equations changes, as described by catastrophe theory (Thom, 1975). In this way, "structures" with discrete boundaries may be formed from continuous models. However, such approaches become impractical for systems with very many degrees of freedom, and therefore cannot address the formation of genuinely complex structures.

More general investigations of self-organization and "chaos" in dynamical systems have typically used simple mathematical models. One approach (e.g., Ott, 1981) considers dissipative nonlinear differential equations (typically derived as idealizations of Navier-Stokes hydrodynamic equations). The time evolution given particular initial conditions is represented by a trajectory in the space of variables described by the differential equations. In the simplest cases (such as those typical for chemical concentrations described by the Boltzmann transport equations), all trajectories tend at large times to a small number of isolated limit points, or approach simple periodic limit cycle orbits. In other cases, the trajectories

https://www.stephenwolfram.com/scrapbook/page3/#1982_discoveries

Statistical Mechanics of Cellular Automata

Cellular automata are sufficiently simple to allow detailed mathematical analysis, yet sufficiently complex to exhibit a wide variety of complicated phenomena. Cellular automata are also of sufficient generality to provide simple models for a very wide variety of physical, chemical, biological, and other systems. The ultimate goal is to abstract from a study of cellular automata general features of "self-organizing" behavior and perhaps to devise universal laws analogous to the laws of thermodynamics. This paper concentrates on the mathematical features of the simplest cellular automata, leaving for future study more complicated cellular automata and details of applications to specific systems. The paper is largely intended as an original contribution, rather than a review. It is presented in this journal in the hope that it may thereby reach a wider audience than would otherwise be possible. An outline of some of its results is given in Wolfram (1982a).

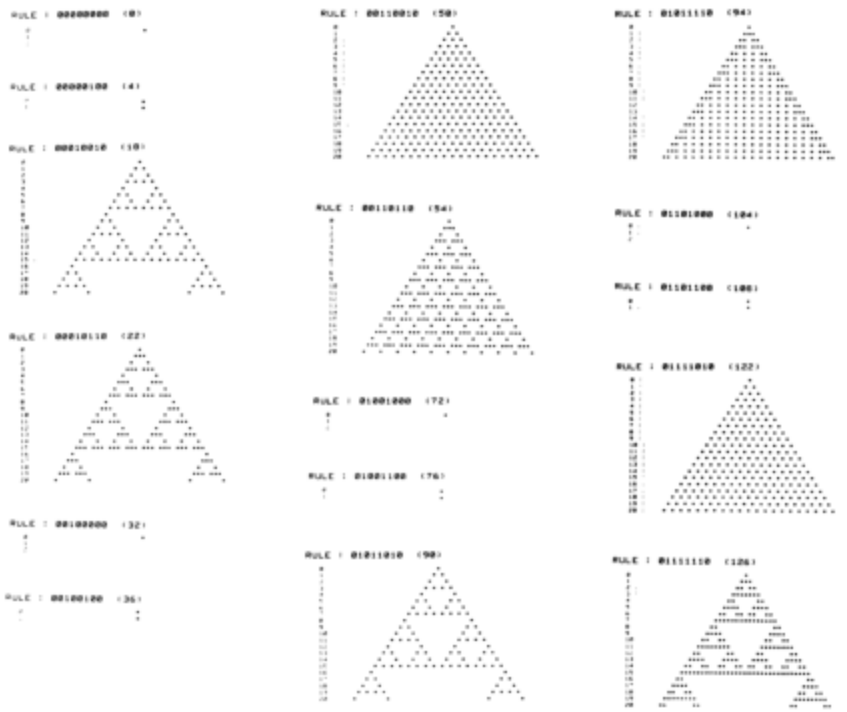

FIG. 3. Evolution of one-dimensional elementary cellular automata according to the 32 possible legal sets of rules, starting from a state containing a single site with value 1. Sites with value 1 are represented by stars, and those with value 0 by blanks. The configurations of the cellular automata at successive time steps are shown on successive lines. The time evolution is shown up to the point where the system is detected to cycle (visiting a particular configuration for the second time), or for at most 20 time steps. The process is analogous to the growth of a crystal from a microscope seed. A considerable variety of behavior is evident. The cellular automata which do not tend to a uniform state yield asymptotically self-similar fractal configurations.

is nontrivial. In the infinite time limit, the configurations are "self-similar" in that views of the configuration with different "magnifications" (but with the same "resolution") are indistinguishable. The configurations thus exhibit the same structure on all scales.

Consider as an example the modulo-two rule 90 (also used as the example for Fig. 1 and in the discussion above). This rule takes each site to be the sum modulo two of its two nearest neighbors on the previous time step. Starting from an initial state containing a single site with value 1, the configuration it yields on successive time steps is thus simply the lines of Pascal's triangle modulo two, as illustrated in Fig. 4 (cf. Wolfram, 1982b). The values of the sites are hence the values of binomial coefficients [or equivalently, coefficients of x^t in the expansion of $(1+x)^n$] modulo two. In the large time limit, the pattern of sites with value 1 may be obtained by the recursive geometrical construction (cf. Sierpinski, 1916; Abelson and diSessa, 1981, Sec. 2.4) shown in Fig. 5. This geometrical construction manifests the self-similarity (Mandelbrot, 1977, 1982; Geffen *et al.*, 1981) or "scale invariance" of the resulting curve. Figure 3 shows that evolution of other complex cellular automata from a single nonzero site yields essentially identical self-similar pat-

Statistical Mechanics of Cellular Automata

Cellular automata are sufficiently simple to allow detailed mathematical analysis, yet sufficiently complex to exhibit a wide variety of complicated phenomena. Cellular automata are also of sufficient generality to provide simple models for a

The generation of self-similar patterns was thus found to be a generic feature of complex cellular automata evolving from simple initial states. This result may provide some explanation for the widespread occurrence of self-similarity in natural systems.

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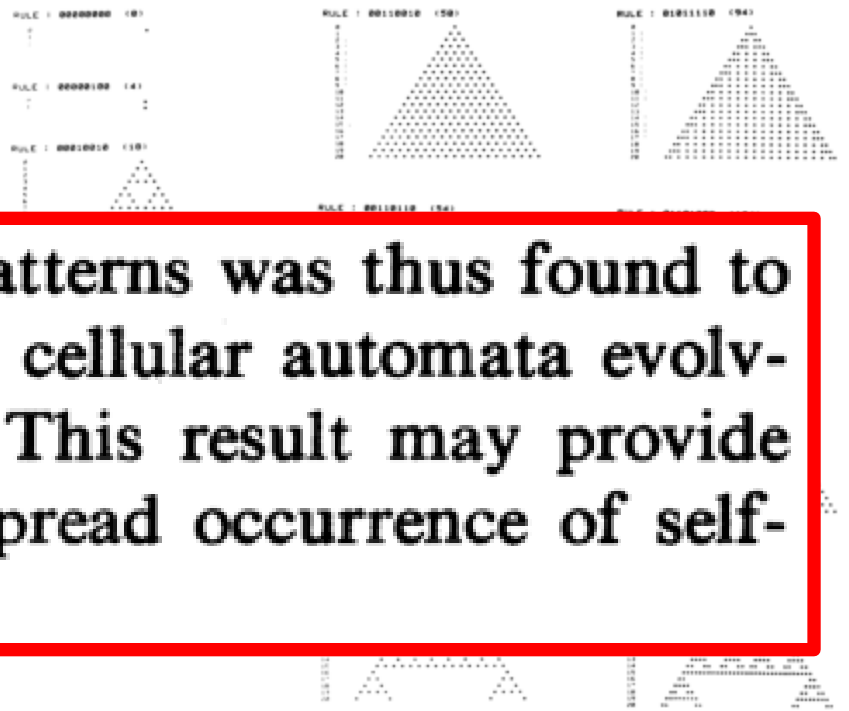

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Statistical mechanics of complex networks

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Complex networks describe a wide range of systems in nature and society. Frequently cited examples include the cell, a network of chemicals linked by chemical reactions, and the Internet, a network of routers and computers connected by physical links. While traditionally these systems have been modeled as random graphs, it is increasingly recognized that the topology and evolution of real networks are governed by robust organizing principles. This article reviews the recent advances in the field of complex networks, focusing on the statistical mechanics of network topology and dynamics. After reviewing the empirical data that motivated the recent interest in networks, the authors discuss the main models and analytical tools, covering random graphs, small-world and scale-free networks, the emerging theory of evolving networks, and the interplay between topology and the network's robustness against failures and attacks.

CONTENTS

I. Introduction 48
II. The Topology of Real Networks: Empirical Results 49
A. World Wide Web 49
B. Internet 50
C. Movie actor collaboration network 52
D. Science collaboration graph 52
E. The web of human sexual contacts 52
F. Cellular networks 52
G. Ecological networks 53
H. Phone call network 53
I. Citation networks 53
J. Networks in linguistics 53
K. Power and neural networks 54
L. Protein folding 54
III. Random-Graph Theory 54
A. The Erdős-Rényi model 54
B. Subgraphs 55
C. Graph evolution 56
D. Degree distribution 57
E. Connectedness and diameter 58
F. Clustering coefficient 58
G. Graph spectra 59
IV. Percolation Theory 59
A. Quantities of interest in percolation theory 60
B. General results 60
1. The subcritical phase ($p<p_c$) 60
2. The supercritical phase ($p>p_c$) 61
C. Exact solutions: Percolation on a Cayley tree 61
D. Scaling in the critical region 62
E. Cluster structure 62
F. Infinite-dimensional percolation 62
G. Parallels between random-graph theory and percolation 63
V. Generalized Random Graphs 63
A. Thresholds in a scale-free random graph 64
B. Generating function formalism 64
1. Component sizes and phase transitions 65
2. Average path length 65
C. Random graphs with power-law degree distribution 66
D. Bipartite graphs and the clustering coefficient 66
VI. Small-World Networks 67
A. The Watts-Strogatz model 67
B. Properties of small-world networks 68
1. Average path length 68
2. Clustering coefficient 69
3. Degree distribution 70
4. Spectral properties 70
VII. Scale-Free Networks 71
A. The Barabási-Albert model 71
B. Theoretical approaches 71
C. Limiting cases of the Barabási-Albert model 73
D. Properties of the Barabási-Albert model 74
1. Average path length 74
2. Node degree correlations 75
3. Clustering coefficient 75
4. Spectral properties 75
VIII. The Theory of Evolving Networks 76
A. Preferential attachment $\Pi(k)$ 76
1. Measuring $\Pi(k)$ for real networks 76
2. Nonlinear preferential attachment 77
3. Initial attractiveness 77
B. Growth 78
1. Empirical results 78
2. Analytical results 78
C. Local events 79
1. Internal edges and rewiring 79
2. Internal edges and edge removal 79
D. Growth constraints 80
1. Aging and cost 80
2. Gradual aging 81
E. Competition in evolving networks 81
1. Fitness model 81
2. Edge inheritance 82
F. Alternative mechanisms for preferential attachment 82
1. Copying mechanism 82
2. Edge redirection 82
3. Walking on a network 83
4. Attaching to edges 83
G. Connection to other problems in statistical mechanics 83
1. The Simon model 83
2. Bose-Einstein condensation 85

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Statistical Mechanics of Networks

- A real review article, covering seminal developments:
  - Small world networks
  - Scale free networks
- But timely and prescient in the scope, implications and future significance of the subject matter

F. Conclusions

The shift that we have experienced in the past three years in our understanding of networks was swift and unexpected. We have learned through empirical studies, models, and analytic approaches that real networks are far from being random, but display generic organizing principles shared by rather different systems. These advances have created a prolific branch of statistical mechanics, followed with equal interest by sociologists, biologists, and computer scientists. Our goal here was to summarize, in a coherent fashion, what is known so far. Yet we believe that these results are only the tip of the iceberg. We have uncovered some generic topological and dynamical principles, but the answers to the open questions could hide new concepts and ideas that might turn out to be just as exciting as those we have encountered so far. The future could bring new tools as well, as the recent importation of ideas from field theory (Burda *et al.*, 2001) and quantum statistics (Bianconi, 2000a, 2001; Bianconi and Barabási, 2001b; Zizzi, 2001) indicates. Consequently this article is intended to be as much a review as a catalyst for further advances. We hope that the latter aspect will dominate.

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- Previews of Modern Physics define a future field of science
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  - Don't be afraid to give away your best ideas in this format, rather than a PRL, Nature Physics, …
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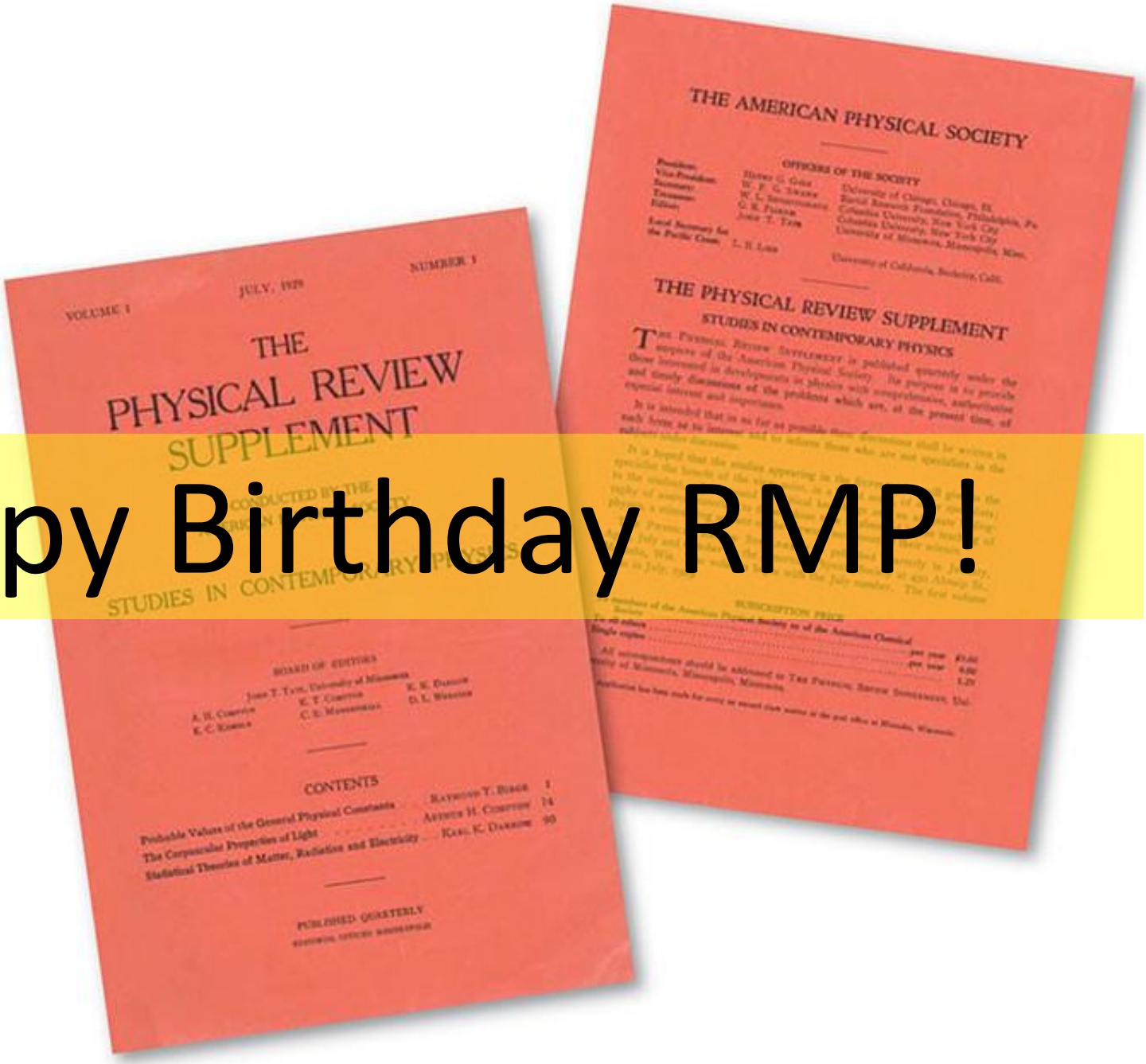
  - They are the first word on a subject, not the last
- Previews of Modern Physics can be wrong, but they fail spectacularly
  - By addressing risky problems, they maximise the chance for discoveries that are deep and subtle



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